THREE ESSAYS IN AUCTIONS AND CONTESTS

by

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ABSTRACT

This thesis studies issues in auctions and contests. The seller of an object and the organizer of a contest have many instruments to improve the revenue of the auction or the efficiency of the contest. The three essays in this dissertation shed light on these issues.

Chapter 2 investigates how a refund policy affects a buyer's strategic behavior by characterizing the equilibria of a second-price auction with a linear refund policy. I find that a generous refund policy induces buyers to bid aggressively. I also examine the optimal mechanism design problem when buyers only have private initial estimates of their valuations and may privately learn of shocks that affect their valuations later. When all buyers are **ex-ante** symmetric, this optimal selling mechanism can be implemented by a first-price or second-price auction with a refund policy.

Chapter 3 investigates how information revelation rules affect the existence and the efficiency of equilibria in two-round elimination contests. I establish that there exists no symmetric separating equilibrium under the full revelation rule and find that the non-existence result is very robust. I then characterize a partially efficient separating equilibrium under the partial revelation rule when players' valuations are uniformly distributed. I finally investigate the no revelation rule and find that it is both most efficient and optimal in maximizing the total efforts from the contestants. Within my framework, more information revelation leads to less efficient outcomes.

Chapter 4 analyzes the signaling effect of bidding in a two-round elimination contest. Before the final round, bids in the preliminary round are revealed and act as signals of the contestants' private valuations. Compared to the benchmark model, in which private valuations are revealed automatically before the final round and thus no signaling of bids takes place, I find that strong contestants bluff and weak contestants sandbag. In a separating equilibrium, bids in the preliminary round fully reveal the contestants' private valuations. However, this signaling effect makes the equilibrium bidding strategy in the preliminary round steeper for high valuations and flatter for low valuations compared to the benchmark model.

CO-AUTHORSHIP

Chapter 3 of this thesis was co-authored with Ruqu Wang in the Department of Economics, Queen's University.

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CHAPTER 1

INTRODUCTION

Refund policies (or money back policies) are widely used in real life. Many large chain stores, such as Hudson's Bay Company, Bestbuy, Sears and Wal-Mart, usually provide full refund policies. With the increasing popularity of auctions, it is now quite common for sellers in auctions to have refund policies. NHL (National Hockey League) online auctions, for example, provide a 7-Day, 100% Money-Back Guarantee. Most sellers in online auctions, such as Amazon.com, eBay.com, Johareez.com and Yahoo.com[1] , provide partial refund policies in which buyers have to pay some fees if they would like to return merchandise. A refund policy is different from a warranty. A warranty insures customers against defective products. It usually involves a pre-specified contractible event, such as the "failure" of a product, and the seller provides some compensation to the buyer in this situation.[2] Refund policies, on the other hand, do not require customers to provide evidence or explanation regarding the malfunctioning of returned goods. Instead, it is usually sufficient if a customer dislikes the product; this is usually referred to a

[1] Yahoo US and Canada Auctions sites were retired on June 16, 2007. This did not impact Yahoo Auction sites in Hong Kong, Singapore and Taiwan.

[2] The literature on warranties is usually discussed in the context of signaling. To name a few, see Grossman [23], Heal [29], Mann and Wissink [41][42], Moorthy and Srinivasan [51] and Wellig [64]. In this literature, sellers have better information about, or sometimes control, the qualities of their own products. As a result, sellers provide different warranties for products with different qualities. The literature on optimal mechanism design when a principle is better informed than agents is named the informed principle problem, see Myerson [53], Maskin and Tirole [43], Severinov [60] and Skreta [61].

"no-question-asked" policy. Consumers return products due to idiosyncratic shocks, which are added to their initial estimates, resulting from matching tastes or styles, complementarity with other products consumers may own, etc. In reality, to maximize profits, a seller should take both defectiveness and idiosyncratic shocks into consideration, and this is why many products come with both warranties and refund policies. In this essay, I focus on refund policies.

To be more concrete, I will embody my analysis in an environment where a seller faces buyers who only have private initial estimates of their valuations and may privately learn their shocks that affect their valuations later.[3] Specifically, a buyer's valuation is the sum of his initial estimate and the shock. At the beginning, buyers only observe their initial estimates privately. A buyer learns his shock only if he gets the object and the realization of the shock is his private information. This environment is especially suitable for online auctions since buyers cannot evaluate items accurately simply from the descriptions provided by the sellers. This assumption applies to many other situations as well. For example, bidders in wholesale flower auctions may not be certain about the demand in their home markets. Bidders for a house may be uncertain about the value of the house resulting from future career prospects in the area, whether the neighborhood is friendly, etc. Bidders in wine auctions may not necessarily know accurate valuations of a bottle of wine until they open the bottle and taste the wine. I ask the following questions. (1) How does a refund policy affect a buyer's bidding strategy in an auction? (2) Is there a reason why or why not a rational seller provides a refund policy? If provide, what is the optimal refund policy?[4] This essay aims to address these questions.

Auctions in Amazon.com, eBay.com, Johareez.com and Yahoo.com are English auc-

[3] In a series paper, Haile [26][27][28] considers a similar environment but in the context of resale. Due to the inefficiency caused by some new information flow, the winner in an auction can resell the object to the losers.

[4] People may say it is hardly to see sellers in house auctions and wine auctions provide refund policies. However, note that providing a very strict refund policy is equivalent to not allowing return.

tions, which are strategically equivalent to second-price auctions under certain conditions. This property is also valid in the setting of this essay. In these auctions, sellers usually charge fees for returns, and the fees are linear functions of the transaction prices. Therefore, to answer the first question, I examine a second-price auction with a linear refund policy. In the return stage, the winner returns the object if and only if his realized valuation is less than the transaction price in the auction minus the cost of return. In the auction stage, this essay finds that providing a refund policy induces buyers to bid more aggressively. The more generous the refund policy is, the more aggressively buyers bid. In addition, if a refund policy is too strict, i.e., the fee is too high, the outcome is equivalent to the outcome when no return is allowed.

To answer the second question, I examine the optimal selling mechanism. First, I find that in the optimal selling mechanism, the seller chooses to sell to the buyer with the highest "modified virtual initial estimate" if it is higher than a cutoff, and allows him to return the object if his realized shock is lower than a cutoff. An auction with a refund policy can represent the optimal selling mechanism. This mechanism is separable. Competition among buyers only affects the selection of the winner. The refund policy depends only on the winner's initial estimate and his shock, not on the number of buyers or losers' types. This means that the seller should provide the same refund policy for the winner as if he were the only buyer. As a result, this mechanism can be divided into two parts. The first issue is how to select the best buyer. Since buyers have private information, it is generally optimal to choose the one with the highest initial estimate. Myerson [52] shows that the winner should be the one with the highest virtual initial estimate or, as interpreted in Bulow and Robers [9], the highest marginal revenue of selling. In my essay, the virtual valuation is modified because of the learning about the shock later on. However, when the distribution of the shock is the same for all the buyers, using Myerson's optimal auction can select the right winner. The second issue is how to design the optimal refund policy. There are two important features of this optimal refund policy. First, there is excessive return for the winner unless his initial estimate is at its upper bound. This distortion is introduced only by the fact that initial estimates are buyers' private information. The fact that buyers may privately learn of shock does not introduce any distortion. Second, the amount of refund is a decreasing function of the winner's initial estimate when the hazard rate functions of the initial estimates are increasing. This suggests that the commonly observed linear refund policy is actually never optimal.

Second, buyers cannot gain any informational rents for their shocks that they may privately learn later. This is because the buyers have no information advantages over the seller regarding their shocks at the contracting stage. Since the seller is proposing the contract and has all the bargaining power, she extracts all the benefit. This result is straightforward in the following example. Suppose there is only one buyer. His initial estimate is always zero and his shock is uniformly distributed on $[0, 1]$. The value of the object to the seller is zero. If the seller can observe the buyer's shock when it is realized, the optimal strategy is to charge him the price equal to the buyer's realized shock, since it always leaves the buyer with zero surplus. As the buyer does not learn his shock at the very beginning, the expected revenue for the seller is 0.5. However, even if the seller does not observe the buyer's shock, she can get the same expected revenue by charging him 0.5 for the object at the very beginning. This mechanism is incentive compatible since the buyer's report about his shock does not affect anything. In addition, the buyer is indifferent between accepting and rejecting the contract.

Chapters 3 and 4 consider contests under incomplete information. Contests are widely used by various organizations wishing to select the best person or the best team. Sports, for example, are undoubtedly always in the form of contests. Science contests in schools are another example. I can consider a contest as an auction in which players compete for prizes by submitting “bids”. These “bids”, once submitted, have to be paid regardless of winning or losing. R&D races, political elections, competitions for promotions within certain organizations, lobbying, etc., all fit in the framework of contests. The “bids” in contests such as R&D races, political elections, lobbying are the amount of money spent by each agent, while in sports, science competitions, and promotion competitions, these “bids” become the amount of effort exerted by each contestant. Since all participants have to pay their bids regardless of winning or not, contests are actually all-pay auctions. In the literature, Bay et al [7], Hillman and Riley [30], and Moldovanu and Sela [50], to name a few, all model contests as all-pay auctions.

Limited by the number of contestants in each contest, many contests are held in multiple stages. Contestants are initially divided into a few groups and competitions within each group are held first. The winners from each group compete again in later stages, and the losers are eliminated from the competitions. As pointed out in Moldovanu and Sela [50], “[b]esides sports, [these] elimination contests are very popular and widely used in the following situations: (1) in the organization of internal labor markets in large firms and public agencies, the sub-contests are usually regional or divisional, and the prizes are promotions to well-defined (and usually equally paid) positions on the next rung of the hierarchy-ladder; (2) in political competition (e.g., for the US presidency), candidates first spend resources to secure their party’s nomination, and later, if they are nominated, spend more resources to get elected; and (3) science contests among university or high-school students, e.g., the Mathematics Olympiad.”

It is commonly believed that these multi-stage contests are capable of selecting the “best” contestants. Indeed, Moldovanu and Sela [50] show that the contestant with the highest willingness to pay for the prize will submit the highest bid in each stage and thus win the contest. Their analysis is based on the assumption that no information is released after each stage of the contest. In many contests, however, the assumption of no information revelation is not applicable. Sometimes, the bids of the contestants may be fully or partially revealed. In sports, for example, the performance of each team in each stage of contest is publicly observed and cannot be concealed. In competition for promotions within a firm, for another example, the performance of different workers can also be partially observed by each other. In some cases, the organizers can even control the amount of information released to the contestants. For example, the U.S. Federal Election Commission can have different policies on whether presidential election candidates should reveal the amount of money they raised during election campaigns. In schools, for another example, school officials can decide whether to reveal the scores of the students in each round of science contests. In this essay, I investigate whether multi-stage elimination contests can still select the best contestant in situations where some information is released in the interim stages.

In Chapter 3, I consider the simplest model possible with two rounds of contests and four ex ante identical players. Each player has some privately informed valuation for the prize offered by the contest. In the first round, the four players are equally divided into two groups. The two players in the same group compete in an all-pay auction; the player with the lower bid is eliminated from the contest and the player with the higher bid becomes the winner in that group. In the second round, the two winners from the first round compete again in another all-pay auction and the player with the higher bid becomes the second round winner of the contest and receives the prize.

In order for the contest to be efficient, that is, the player with the highest valuation always wins the prize, it must be the case that the higher valuation player submits a higher bid in each auction. With my assumption that players are ex ante identical, contest efficiency can be achieved only when the players use a symmetric and strictly increasing bidding function in each auction. I find that when players' bids in the first round are publicly revealed before the second round competition, there exists no symmetric strictly increasing equilibrium bidding function, and therefore the player with the highest willingness to pay may not always win the prize. Hence, multi-stage elimination contests with full revelation of bids are not efficient.

The intuition behind the above result that multi-stage elimination contests with full revelation of bids fail to always select the player with the highest willingness to pay (valuation) as the winner is as follows. In an efficient equilibrium where the player with the highest willingness to pay always wins, players must use a symmetric strictly increasing bidding function in each of the two rounds of the contest. That means the equilibrium is completely separating. With full revelation of bids right after the first round, players in the second round have complete information regarding each other's valuation along the equilibrium path. However, a player's expected payoff calculated at the start of the first round contest is not differentiable in his bid at the value of his equilibrium bid. This is because to ensure that a player does not want to pretend to have a lower valuation, the bidding function in the first round needs to be relatively flat; conversely, in order to ensure that a player does not want to pretend to have a higher valuation, that bidding function needs to be relatively steep. As a result, no bidding function can satisfy the incentive compatibility constraints in both directions in the first round. Therefore, if an equilibrium exists, it must involves either non-monotone bids, pooling of bids, or mixed strategies in the first round. This implies that the winners in the first round contests may not have the highest valuation and entries to the second round contest may not be efficient.

One may wonder if the non-existence result is very sensitive to the specifications of the model. My analysis shows that it is quite robust. In my two-round elimination model, even if I allow for interim prizes for the first round winners, the non-existence result remains valid. The result is still true when a player's valuation for the final prize changes by a random amount in the second round.

I also investigate other information revelation rules in this essay. One rule I investigate is the partial information revelation rule. Under this rule, the losers' bids are revealed after the first round competition. This could be the case when the organizer of the contest cannot control the behavior of those who are eliminated. Losers from multi-round, sealed-bid auctions, for example, can always reveal their failed bids to the public. Another rule I investigate is the no information revelation rule. No information other than the identity of the winners and losers will be released before the entire contest is over.

Under the partial revelation rule, I am able to characterize a separating symmetric Perfect Bayesian Nash Equilibrium under the assumption that the contestants' valuations follow an i.i.d. uniform distribution. The efficient entries to the second round are achieved. However, the final allocation of the prize is not efficient since the finalists are in an asymmetric contest in the second round.

In a game of incomplete information, there is no doubt that information revelation rules play a crucial role in players' strategies. They affect players' beliefs about each other's type before they compete in the second round. As a result, these rules change the game dramatically.[5] A natural question arises. If the information revelation rule is a choice for the contest designer who wants to maximize the total efforts (bids) of the players, how should he choose

[5] Early work on eliminating contests considers the case of complete information. See Groh et al [22], Horen and Riezman [31], Hwang (1982) [35], Rosen [58], and Schwenk [59].

the information revelation rules optimally? After the first round bidding is over, he can make all kinds of announcements, such as announcing all of the bids, announcing no bids, announcing some of the bids, announcing the mean, the average of the bids, or even announcing bids stochastically. It is impossible to formulate all possible information revelation rules. I show that the no information revelation rule, in addition to being efficient, is also optimal among all information revelation rules for a contest designer who has to award the prize to a player and who wants to maximize the total bids of the players.

Based on my results, more information revelation in the interim stage of certain dynamic games leads to less efficient outcome. This conclusion suggests that it is more efficient for a firm not to reveal information of those employees competing for a promotion, that it is more efficient for school officials to keep the scores of contestants secret in various stages of science competition, and that privacy laws improve the efficiency of my society. Of course, the analysis and the models are somewhat restricted, especially under the rule of partial information revelation in my model, and therefore this conclusion needs to be interpreted with caution.

One unpleasant result in Chapter 3 is the non-existence of a symmetric equilibrium under full information revelation, since I can not talk about how the signalling effect works. In Chapter 4, I change the model in Chapter 3 and assume that the preliminary round is an all-pay auction and that the final round is a lottery.[6] In this case the existence of a symmetric separating equilibrium is restored. Contestants have incentives either to engage in "bluffing" by submitting a bid higher than their true types would submit in equilibrium or to engage in "sandbagging" by submitting a bid lower than their true types would submit in equilibrium in order to gain some advantages in the later round.

[6] Hurely and Shogren [33, 34], Nti [54], Tullock [62] model contests as lotteries where a player's winning probability is equal to the ratio of his own bid to the total bids.

In a separating equilibrium, players' actions (bids) in the preliminary round fully reveal their valuations, and there is complete information in the final round. Generally speaking, a bid in the preliminary round has two effects. It changes a player's winning probability in the preliminary round, and it also has a signaling effect. Since I want to investigate how the signaling effect affects a player's strategy, it is natural to look at the benchmark model in which players' valuations are revealed automatically and become common knowledge before the final round. This benchmark excludes the signaling effect of a bid while preserving its effect on the winning probability.

Comparing this benchmark with the original model fully characterizes the signaling effect. This essay finds that weak contestants sandbag and strong contestants bluff in the preliminary round in the presence of the signaling effect.[7] In a separating equilibrium, a bid in the preliminary round fully reveals a contestant's private valuation. However, this signaling effect imposes a downward pressure on the equilibrium bidding strategy of weak contestants and an upward pressure on the equilibrium bidding strategy for strong contestants in the preliminary round. In other words, this signaling effect makes the equilibrium bidding strategy in the preliminary round steeper for high valuations and flatter for low valuations compared to the benchmark. Intuitively, the signaling effect works as follows. Since a player's expected valuation of entering the final round is a decreasing function of his rival's bid in the final round, there is an incentive for players to disguise their own valuations in order to reduce their rivals' bids in the final round. Generally speaking, if a player is strong, he anticipates that he will have a greater chance to meet a player weaker than himself in the final round. As a result, he bluffs in the preliminary round since he wants to discourage his rival in the final round. If he is a weak

[7] Horner and Sahuguet [32] state "bluffing (respectively sandbagging) occurs when a weak (respectively strong) player seeks to deceive his opponent into thinking that he is strong (respectively weak)."

player, he anticipates that he will have a greater chance to meet a player stronger than himself in the final round. As a result, he sandbags in the preliminary round since he would want his rival to underestimate him in the final round.

In this essay, a separating equilibrium may fail to exist due to weak players' sandbagging in the preliminary round. This happens when the signaling effect becomes strong enough to dominate the effect that pretending to be weaker decreases the winning probability. In this case, it is always better for a player to under-represent his valuation in the preliminary round, since the gain increases while the bidding cost decreases. I am able to identify a sufficient condition to guarantee the existence of a separating equilibrium.

A salient feature of this model is that the outcome in the preliminary round (all-pay auction) is more sensitive to the bids than that in the final round (lottery). The reason is the tractability. First, if I model both rounds as lotteries, then at the beginning of the preliminary round, all players face symmetric incomplete information. However, there is no tractable solution for lottery under incomplete information in general.[8] Second, if I model both rounds as all-pay auctions, then, as shown in Chapter 3, there exists no separating equilibrium. Third, it is well known in the contest literature that if the winning probability in the lottery is determined by $\frac{b_k^\lambda}{\sum_{i=1}^{I} b_i^\lambda}$, where b_i is player $i's$ bid and λ is known as the sensitivity of the administrators, then an all-pay auction is equivalent to a lottery when λ goes to infinity. Therefore, I am actually modelling the two rounds with lotteries but with different sensitivity of the administrators.

Although the main reason for this heterogeneity is the tractability, it is not vacuous in real life elimination contests. In the USA, success in the primaries is more sensitive to effort than success in the presidential election since the latter is prone to more noise. In NHL or NBA

[8] Several papers in the literature, Hurely and Shogren [33, 34], and Malueg and Yates [40], have been written under special settings, such as discrete private information or one-sided asymmetric information.

tournaments, the regular season have many more games and therefore the teams entering the playoffs are usually the better teams. In contrast, the playoffs have less games and the result is more randomness. Many famous TV shows also fit my model.[9] For example, in “American Idol”, the contestants compete in their own divisions for tickets to Hollywood and the winners compete again in the final round.[10] In the preliminary rounds, it is only the judges’ votes that count. Given that the judges are experts, it is very likely that the result in this round will be very sensitive to effort and, as a result, it is the singers with more talent who win, similar to an all-pay auction. When winners get to Hollywood, the votes of the judges no longer count and only the votes of viewers count. Since viewers are not experts, it is possible that success may not be sensitive to effort in this stage and the final round becomes a lottery.[11]

[9] I am grateful for James Amegashie for alerting me of this example

[10] My model is a simple version of the ”American Idol”. In my model, there is only one winner in the preliminary round; in contrast, in the “American Idol”, there are multiple tickets to go to the Hollywood in the preliminary round

[11] Indeed, in the recent season of “American Idol”, there was a contestant with bad performance, Sanjaya, who the judges did not like must kept advancing in the final round because the viewers liked him.

CHAPTER 2

AUCTIONS WITH REFUND POLICIES AS OPTIMAL SELLING MECHANISMS

2.1 Related literature

This essay is related to Courty and Li [12].[1] At the time of contracting, consumers only privately learn their types and subsequently learn their actual valuations. They find that it is optimal for the seller to provide a menu of refund contracts to sequentially screen consumers. In their paper, the seller has a production function with a constant marginal cost and she can serve the whole market. Therefore, there is no competition among consumers. It is equivalent to a model where a seller contracts with a single representative consumer. In contrast, in my essay there is only a single indivisible object, and the selection of the winner plays an important role. Introducing competition among consumers makes it difficult to handle the direct mechanisms. In my essay, after all buyers announce their initial estimates in the first stage, the seller has to control what kind of information she would like to reveal regarding those reports. This information revelation within stages affects the winner's incentive compatibility constraint in the second stage if either second stage allocation rules or payments depend on the losers' reported initial estimates in the first stage. However, it is complicated to formulate all the possible infor-

[1] Akan et al. [1] extend their model to allow two dimensions of potential asymmetry across consumers: **ex-ante** distribution of valuations and the time at which consumers learn their valuations.

mation revelation rules within stages to find the optimal mechanism. This difficulty is absent in Courty and Li [12], since there is only a single representative consumer and he always has perfect information in the second stage regardless of the information revelation. Their model can be solved by formulating all the direct mechanisms to find the optimal mechanism.

The result that buyers cannot gain any informational rents for their private information realized after the contract is signed is similar to the findings in several papers. Eso and Szentes [19] consider the situation in which a seller faces buyers with initial estimates of the object and she can control and costlessly release additional private signals which are correlated to buyers' valuations. Because the release of information is costless for the seller, she releases all the information to all the buyers. In this essay, the seller does not control the releasing of the shocks. The winner learns his shock privately because of the nature of the model. In addition, the seller needs to select one buyer who can acquire additional information. Note that the method to solve my model is inspired by their paper and I will discuss this method in detail in Section 5. Cremer et al. [15] consider the environment where bidders must incur a cost to learn their valuations and the participation constraints are **ex-ante**.[2] In their paper, since bidders do not have private information before the contract is signed, the seller extracts all the surplus by inducing an efficient allocation. In my essay, the optimal mechanism generally does not induce an efficient allocation, since part of buyers' private information is realized before the contract is proposed. In Eso and Szentes [20], a consultant can reveal signals that affect her client's valuation and can make the payment rule conditional on the client's action. In their paper, it is the third party, the consultant, who controls the signal. Board [8] analyzes selling dynamic options to competing buyers with payments possibly conditional on exercising

[2] Cremer et al. [14] change the model slightly by proposing interim participation constraints and assume that the seller incurs costs to contact prospective bidders.

the option (where the exercising decision is subject to moral hazard). Pavan et al. [56] derive a seller's optimal mechanism for multi-period contracting, extending the result of Myerson's optimal auction to a dynamic setting.

This essay is related to the literature on refunds (or money back guarantees). Che [11] considers consumers' risk aversions and obtains very intuitive results. A refund policy provides an insurance for a consumer's **ex-post** loss. Since the buyer is risk averse and the seller is risk neutral, the seller gains from the risk premium. However, in order to induce the seller to provide refund policies, consumers must be highly risk averse. Furthermore, providing a refund policy can never be optimal when buyers are risk neutral in his paper. In my essay, the seller is better off providing refund policies even if she is facing risk neutral buyers. Davis et al. [16] find that when the salvage value of a product is relative large, the seller gains from refund policies, and this is consistent with my finding. Both papers only compare the selling with the full refund policy with the no refund policy and compute the conditions under which the full refund policy is better. In addition, they assume that there is only one representative consumer. In contrast, I compute the optimal refund policy and there are multiple consumers in my model.

This essay is related to the literature on dynamic mechanism design.[3] Deb [17] considers the optimal contracting of new experience goods using a T-period model. A consumer's valuation is uncertain and he receives an additional signal in a period whenever he consumes in that period. Pavan [55] analyzes a seller's optimal long-term contract with an agent whose type changes over time. The seller has full commitment power, while the buyer makes participation decision in every period. Battaglini [6] considers a model with one agent and two types and

[3] The majority of the literature on mechanism design focuses on static models in which contracting occurs only in a single period and assumes that the agents know their valuations at the time at which they agree to the contract, to name a few, see Armstrong [4], Eso [18], Laffont and Robert [37], Maskin and Riley [45], and Myerson [52].

derives an optimal selling mechanism for a monopolist facing a consumer whose type follows a Markov process. Since only a single agent is considered, the selection of the winner is absent. Introducing competitions among buyers is not trivial since direct mechanisms are now much more complicated.

In a different approach, Matthews and Persico [46] show that a refund policy is optimal when consumers can costly acquire some information to learn their true valuation before contracting. They propose two different models. In the model of information acquisition, the refund policy weaken consumers' incentive to acquire information before purchasing and therefore eliminates the informational rents. In the model of screening, by offering a refund, the firm is able to charge a higher price, and it chooses to do this if the price it would like to charge the informed consumers in isolation is sufficiently high. In their paper, it is the consumer who controls the information acquisition.

2.2 Environment

One seller, possessing full commitment power, with a single indivisible object faces n buyers. Let N represent the set of buyers, i.e. $N = \{1, \cdots, n\}$. All parties are risk neutral. Buyer i's valuation of the object V_i is composed of two parts, $W_i + \Phi_i$. I refer to W_i as his initial estimate and Φ_i as his shock. I assume that W_i and Φ_i are independent to each other and also across i. This is a standard assumption to rule out full information extracting discussed in Cremer and Mclean [13]. The distribution of buyer i's initial estimate is F_i, with associated density function f_i and support $\mathcal{W}_i = [\underline{w}_i, \overline{w}_i]$. The distribution of buyer i's shock is G_i, with associated density function g_i and support $\oplus_i = [\underline{\phi}_i, \overline{\phi}_i]$. I assume that all shocks have zero means,[4] and that the worst valuation is nonnegative for any buyer i, i.e. $\underline{w}_i + \underline{\phi}_i \geq 0, \ \forall i \in N$, which means the object is always desirable for everyone. Note that given Φ_i with zero mean, its lower bound is negative and its upper bound is positive. I assume that the hazard rate function of buyer i's initial estimate $\frac{f_i(w_i)}{1-F_i(w_i)}$ is increasing in w_i in order to simplify the characterization of the optimal mechanism. Uniform, normal, logistic, chi-squared, exponential and Laplace distributions all satisfy this property, see Bagnoli and Bergstrom [5].

I let $\mathcal{W}$ and $\oplus$ denote the sets of all possible combinations of buyers' initial estimates and shocks, i.e.$\mathcal{W} = \mathcal{W}_1 \times \cdots \times \mathcal{W}_n$, $\oplus = \oplus_1 \times \cdots \times \oplus_n$. I let $\mathcal{W}_{-i}$ and $\oplus_{-i}$ denote the sets of all possible combinations of initial estimates and shocks which might be held by buyers other than i, so that $\mathcal{W}_{-i} = \times_{j \in N, j \neq i} \mathcal{W}_j$, $\oplus_{-i} = \times_{j \in N, j \neq i} \oplus_j$.

2.3 Second-price auctions with linear refund policies

This section shows how a refund policy affects buyers' bidding strategies in auctions. The information structure is as follows. At the beginning of an auction, buyer i privately observes his initial estimate w_i; nobody can directly observe any of the shocks. Then a second-price auction is held. Ties are broken by a fair coin toss. At the end of the auction, the winner pays the second highest bid, say p, and the seller sends the object to the winner. Next, the winner, say buyer i, privately learns his realized shock, ϕ_i, and decides whether to keep the object or to return it to the seller. If he keeps the object, the game ends. If he returns it, he has to pay the seller a linear fee, $Kp + C_r$, where $0 \leq K \leq 1, \ C_r \geq 0$, and gets back what he has paid in the auction p. In addition, he incurs a cost C_b to send the object back to the seller. Define $C = C_r + C_b$. I assume that the seller receives an exogenous salvage value of the object S if

[4] If one's shock does not have zero mean, I can always redefine his initial estimate since all buyers are risk neutral.

the winner returns the object.[5] This assumption can be justified as follows. First, if the object is not in its original package, the seller needs to send the returned object to the manufacturer, or sells it as an open box or refurbished item, and this changes the value of the object. Second, after the winner returns the object, market conditions may have already changed when the seller wants to resell the object, since she can no longer find those buyers in the previous auction. Riordan and Sappington [57] make a similar assumption. Using their logic, it is possible that the seller finds it optimal to send the object to only one of the buyers if the cost of sending the object to a buyer is high.

If the seller does not provide a refund policy, the game is equivalent to a standard second-price auction and it is a weakly dominant strategy to bid one's expected valuation of the object. Here, buyers' expected valuations turn out to be their initial estimates since shocks have zero means. If the seller does provide a refund policy, then a buyer anticipates that he can return the object for a refund if his realized shock is too low. This game consists of two stages: the auction stage and the return stage. I focus on the Perfect Bayesian Nash Equilibrium(PBNE) with a weakly dominant strategy in the auction stage. I solve the model by backward induction.

First consider the return stage. If the winner, say buyer i, keeps the object, he receives $w_i + \phi_i$. If he returns the object, he gets back what he has paid in the auction p, minus the total cost for return, $Kp + C$. Since p is the second highest bid in the auction, the decision of whether to return the object or not does not depend on the winner's report in the auction stage. This simplifies the analysis since a buyer has no incentive to manipulate his report in the auction stage to gain an advantage in the return stage. It is quite straightforward to characterize the winner's strategy in the return stage. This is presented in the following lemma.

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[5] In Zhang [65], I consider the case in which the seller can resell the object to other buyers in the auction and the salvage value is endogenously determined.

Lemma 1 In the return stage, the winner keeps the object if $w_i + \phi_i \geq (1-K)p - C$ and returns it if $w_i + \phi_i < (1-K)p - C$.

Here I implicitly assume that if the winner is indifferent between keeping and returning the object, he chooses to keep it. This does not affect any of the results. Now consider the auction stage. I look for each buyer's weakly dominant strategy in this stage, which is given by the following proposition.

Proposition 1 It is a weakly dominant strategy for buyer i to bid b_i^* in the auction stage. The value b_i^* is given by the solution to the following equation,

$$\int_{(1-K)b_i^*-C-w_i}^{\overline{\phi}_i} (w_i + \phi_i - b_i^*) dG_i(\phi_i) - \int_{\underline{\phi}_i}^{(1-K)b_i^*-C-w_i} (Kb_i^* + C) dG_i(\phi_i) = 0. \quad (2.1)$$

In a standard second-price auction, it is a weakly dominant strategy to bid one's highest willingness to pay. In other words, it is a weakly dominant strategy to bid an amount such that he receives zero expect surplus if he pays that amount to the seller for the object. Here the left hand side of equation (2.1) is exactly buyer i's expect surplus if he pays b_i^* to the seller for the object. The first term is his payoff if he keeps the object and the second term is his payoff if he returns it. Therefore, b_i^* defines buyer i's highest willingness to pay.

To understand how a refund policy affects a buyer's bidding strategy, it is necessary to discuss how K and C affect b_i^*.

Corollary 1 If $K = C = 0$, then $b_i^* \geq w_i + \overline{\phi}_i$; if $Kw_i + C \geq -\underline{\phi}_i$, then $b_i^* = w_i$; if otherwise, b_i^* is strictly decreasing in both K and C.

This means providing a refund policy induces buyers to bid aggressively. This is intuitive. A buyer with a bad shock, i.e., he later finds that his valuation is lower than expected, has the option of returning the object. Therefore, the expect value of winning the auction is higher when a refund policy is provided and buyers will bid more. The more generous the refund policy is, the more aggressively the buyers bid.

When there is no cost of getting a refund, i.e. $K = C = 0$, it is always weakly better to win than to lose in the auction. This is because if it turns out that the deal is no good, one can always return the object without any “penalty”. In other words, it does not hurt to buy the whole store and then return everything. This makes the auction very competitive. There are a continuum of weakly dominant strategies for each buyer. Even for an infinitesimal cost, the equilibrium becomes unique. This is good news. In reality, when consumers want to return something, it usually involves some cost, which means C_b is usually greater than zero. Therefore, we do not need to worry about the multiple equilibria problem in practice. When the total cost for return is too high, buyers just discard the opportunity of getting a refund and behave exactly the same way as in a standard second-price auction without a refund policy. Hence, in this case the seller's revenue is the same as that in a standard second-price auction without a refund policy. Therefore, when the seller is allowed to use any refund policy, her revenue will not decrease at the very least, and may even increase.

Potentially, refund policies do not have to be in forms of linear fees. They can take much more general forms. For example, they can be any functions of the transaction price or even buyers' bids in the auction. How to design the optimal selling mechanism and the optimal refund policy is the task of the rest of this chapter.

2.4 The optimal selling mechanism

I start with a description of the abstract environment. At the beginning, all buyers privately observe their initial estimates. No buyer has any information regarding the shocks. Not surprisingly, since buyers privately learn their initial estimates before the contract is signed, they can earn some informational rents. If buyers never learn their shocks later, the environment is equivalent to that in Myerson [52] with the value estimates in his paper replaced by the initial estimates in this essay. Depending on the messages sent by the buyers, the seller first needs to choose one winner to sell the object to and how the first round money transfers are made. The reservation value of the seller is publicly known as R if she keeps the object. If there is a winner, the seller incurs a cost, C_s,[6] to send the object to him. The winner learns his shock. Depending on the new messages, the seller decides whether the object should be returned and how the second round money transfers are made. If the winner returns the object, he incurs a cost C_b,[7] and the seller receives an exogenous salvage value of the object S. It is reasonable to assume that $S \leq R$.

The revelation principle allows us to restrict attention to direct mechanisms when searching for the optimal mechanism. However, even the direct mechanisms are difficult to characterize. After all buyers announce their initial estimates in the first stage, the seller has to control what kind of information she would like to reveal regarding those reports. This information revelation within stages affects the winner's incentive compatibility constraint in the second stage if either second stage allocation rules or payments depend on the losers' reported initial estimates in the first stage. However, it is complicated to formulate all the possible information

[6] This cost could be the transaction cost for example.
[7] This cost could be the disutility of sending the object back or the shipping cost.

revelation rules within stages to find the optimal mechanism. This difficulty can be resolved by using the following general method, which is inspired by Eso and Szentes [9]. First look at a class of two-stage direct mechanisms with full information revelation within stages, i.e., all the reports about initial estimates are made public, and compute the optimal mechanism among this class, called the optimal two-stage direct mechanism. Then, it can be shown that this optimal two-stage direct mechanism is optimal in a relaxed environment, where the seller can observe the winner's realized shock and is able to make her selling mechanism contingent on the winner's realized shock. Obviously, the seller can generate weakly more revenue in the relaxed environment than that in the original environment since she can always ignore the information about the winner's realized shock. Therefore, I can conclude that the optimal two-stage direct mechanism is optimal in the original environment. I begin with solving the optimal two-stage direct mechanism.

2.4.1 The optimal two-stage direct mechanism

The class of two-stage direct mechanism with full information revelation within stages is as follows. The seller has full commitment power on the mechanism. In the first stage, all buyers are asked to report their initial estimates $\tilde{w} = (\tilde{w}_1, \cdots, \tilde{w}_n)$. Then the seller, incurring a cost C_s, sends the object to buyer i with probability $x_i(\tilde{w})$, called buyer i's winning probability, and a first round money transfer $t_i^1(\tilde{w})$ is made from buyer i to the seller.[8] Let $x(\tilde{w}) = (x_1(\tilde{w}), \cdots, x_n(\tilde{w}))$ and $t^1(\tilde{w}) = (t_1^1(\tilde{w}), \cdots, t_n^1(\tilde{w}))$ be the vectors capturing all the buyers' winning probabilities and first round money transfers. At the beginning of the second stage, all buyers' reports about their initial estimates in the first stage are publicly announced. If it

[8] Here I allow for the possibility that a buyer might have to pay something even if he is not the winner.

turns out that the winner is buyer i, he privately learns his shock ϕ_i and reports $\tilde{\phi}_i$ back to the seller. Then, with probability $y_i(\tilde{w}, \tilde{\phi}_i)$, called the winner's keeping probability, the winner keeps the object; with probability $1 - y_i(\tilde{w}, \tilde{\phi}_i)$ the winner, incurring a cost C_b, returns it. A second round money transfer $t_i^2(\tilde{w}, \tilde{\phi}_i)$ is made only from the winner to the seller.[9] Let $y(\tilde{w}) = (y_1(\tilde{w}, \tilde{\phi}_i), \cdots, y_n(\tilde{w}, \tilde{\phi}_i))$ and $t^2(\tilde{w}, \tilde{\phi}_i) = (t_1^2(\tilde{w}, \tilde{\phi}_i), \cdots, t_n^2(\tilde{w}, \tilde{\phi}_i))$ be the vectors capturing all the winners' keeping probabilities and second round money transfers. I call $x_i(\tilde{w})y_i(\tilde{w}, \tilde{\phi}_i)$ buyer i's consuming probability conditional on his shock.

The seller maximizes her revenue by choosing the mechanism (x, y, t^1, t^2). The mechanism must satisfy the incentive compatibility constraints in both stages and the participation constraints in the first stage when the contract is signed. Since there is only one unit of the object to be allocated, the following must be satisfied:

$$\sum_{i=1}^{n} x_i(w) \leq 1 \; and \; x_i(w) \geq 0, \qquad \forall i \in N, \forall w \in \mathcal{W}, \tag{2.2}$$

$$y_i(w, \phi_i) \leq 1 \; and \; y_i(w, \phi_i) \geq 0, \qquad \forall i \in N, \; \forall w \in \mathcal{W}, \; \forall \phi_i \in \oplus_i. \tag{2.3}$$

Define the following functions to simplify the algebra:

$$Q_i(w_i, \phi_i) = \int_{\mathcal{W}_{-i}} x_i(w_i, w_{-i}) y_i(w_i, w_{-i}, \phi_i) dF_{-i}(w_{-i}), \tag{2.4}$$

$$X_i(w_i) = \int_{\mathcal{W}_{-i}} x_i(w_i, w_{-i}) dF_{-i}(w_{-i}). \tag{2.5}$$

Here $Q_i(w_i, \phi_i)$ is buyer i's expected consuming probability conditional on his shock, and $X_i(w_i)$ is his expected winning probability. Since there are two stages in the model, I examine the incentive compatibilities going backwards, starting in the second stage of the mechanism.

[9] Potentially, the seller could ask other buyers to pay as well. However, this does not improve the seller's revenue since all parties are risk neutral. This assumption is made only because asking the losers to pay when the winner returns the object is not observed in practice.

2.4.1.1 The second stage

At the beginning of the second stage, all the reports about initial estimates are made public. Without loss of generality, let buyer i be the winner in the first stage. Given that all the other buyers have truthfully reported their initial estimates in the first stage, I let $\tilde{U}^i(\tilde{\phi}_i, \phi_i; \tilde{w}_i, w_i, w_{-i})$ denote buyer i's payoff in the second stage if he has reported his initial estimate as $\tilde{w}_i$ in the first stage and reports his shock as $\tilde{\phi}_i$ in the second stage. Formally,

$$\begin{aligned}\tilde{U}^i(\tilde{\phi}_i, \phi_i; \tilde{w}_i, w_i, w_{-i}) &= (w_i + \phi_i) y_i(\tilde{w}_i, w_{-i}, \tilde{\phi}_i) - C_b \left[1 - y_i(\tilde{w}_i, w_{-i}, \tilde{\phi}_i)\right] \\ &\quad - t_i^2(\tilde{w}_i, w_{-i}, \tilde{\phi}_i) \\ &= (w_i + \phi_i + C_b) y_i(\tilde{w}_i, w_{-i}, \tilde{\phi}_i) - C_b - t_i^2(\tilde{w}_i, w_{-i}, \tilde{\phi}_i).\end{aligned} \tag{2.6}$$

If he keeps the object, he gets the utility from consuming the object; if he sends back the object, he has to pay the cost C_b; and the money transfer is deducted from his payoff. I first look at the case in which the winner, buyer i, actually has truthfully reported his initial estimate in the first stage. Thus his payoff (2.6) becomes:

$$\tilde{U}^i(\tilde{\phi}_i, \phi_i; w_i, w_i, w_{-i}) = (w_i + \phi_i + C_b) y_i(w_i, w_{-i}, \tilde{\phi}_i) - C_b - t_i^2(w_i, w_{-i}, \tilde{\phi}_i). \tag{2.7}$$

His incentive compatibility constraint in the second stage requires that it is optimal for him to report his realized shock truthfully if he has truthfully reported his initial estimate in the first stage, i.e.,

$$\tilde{U}^i(\phi_i, \phi_i; w_i, w_i, w_{-i}) \geq \tilde{U}^i(\tilde{\phi}_i, \phi_i; w_i, w_i, w_{-i}), \quad \forall i \in N, \forall w \in \mathcal{W}, \forall \phi_i, \tilde{\phi}_i \in \oplus_i \;\; (IC2). \tag{2.8}$$

No participation constraint is needed in the second stage since buyers cannot opt out as long as they have signed the contract in the beginning. The following Lemma simplifies the incentive compatibility constraints in the second stage.

Lemma 2 The incentive compatibility constraints (2.8) in the second stage after truthful reporting of the initial estimate in the first stage are satisfied if and only if the following conditions hold:

$$if\ \phi_i \leq \tilde{\phi}_i,\ then\ y_i(w,\phi_i) \leq y_i(w,\tilde{\phi}_i), \quad \forall i \in N, \forall w \in \mathcal{W}, \forall \phi_i, \tilde{\phi}_i \in \oplus_i, \tag{2.9}$$

$$\begin{aligned} \tilde{U}^i(\phi_i,\phi_i;w_i,w_i,w_{-i}) &= \tilde{U}^i(\underline{\phi}_i,\underline{\phi}_i;w_i,w_i,w_{-i}) \\ &+ \textstyle\int_{\underline{\phi}_i}^{\phi_i} y_i(w,\xi)d\xi, \quad \forall i \in N, \forall w \in \mathcal{W}, \forall \phi_i \in \oplus_i. \end{aligned} \tag{2.10}$$

The proof is quite standard and omitted here. The incentive compatibility constraints require that the winner with a higher realized shock keeps the object more often and his second stage informational rent is determined by his keeping probability. Equation (2.10) also implies that the winner's second round money transfer is determined by his keeping probability. To see this, from equation (2.7) evaluating $\tilde{\phi}_i$ at ϕ_i and equation (2.10), I can pin down the second round money transfer.

Corollary 2 The second round money transfer is given by:

$$\begin{aligned} t_i^2(w,\phi_i) &= (w_i+\phi_i+C_b)y_i(w,\phi_i) - \textstyle\int_{\underline{\phi}_i}^{\phi_i} y_i(w,\xi)d\xi \\ &- C_b - \tilde{U}^i(\underline{\phi}_i,\underline{\phi}_i;w_i,w_i,w_{-i}), \quad \forall i \in N, \forall w \in \mathcal{W}, \forall \phi_i \in \oplus_i. \end{aligned} \tag{2.11}$$

In order to characterize the incentive compatibility constraints in the first stage, I need to look at another continuation in the second stage. I need to know if the winner has lied about his initial estimate in the first stage, how he should manipulate his report about his shock in the second stage. If the winner, say buyer i, misreports his initial estimate in the first stage as $\tilde{w}_i$, his payoff is given by (2.6). The following lemma shows the winner's best response in the second stage.

Lemma 3 In the second stage after misreporting his initial estimate in the first stage, it is optimal for the winner to report his shock $\tilde{\phi}_i$ as $\tilde{\phi}_i^*$ such that:

$$\tilde{\phi}_i^* = w_i + \phi_i - \tilde{w}_i. \tag{2.12}$$

Furthermore, his payoff $\tilde{U}^i(\tilde{\phi}_i^*, \phi_i; \tilde{w}_i, w_i, w_{-i})$ satisfies:

$$\tilde{U}^i(\tilde{\phi}_i^*, \phi_i; \tilde{w}_i, w_i, w_{-i}) = \tilde{U}^i(\tilde{\phi}_i^*, \tilde{\phi}_i^*; \tilde{w}_i, \tilde{w}_i, w_{-i}). \tag{2.13}$$

If the winner misreports his initial estimate in the first stage, he will correct his lie in the second stage in the sense that he will choose to report a shock such that the sum of the reported initial estimate and the shock is equal to his true valuation. Furthermore, his payoff is equal to the payoff of an honest buyer with an initial estimate $\tilde{w}_i$ and a realized shock $\tilde{\phi}_i^*$. The intuition behind the result is as follows. Given that the winner, buyer i, has reported his initial estimate as $\tilde{w}_i$, his problem of how to report his shock is the same as if he had initial estimate $\tilde{w}_i$ and realized shock $\tilde{\phi}_i^*$. The second stage incentive compatibility constraints ensure that truthfully reporting one's shock is optimal if he has truthfully reported his initial estimate in the first stage. Therefore, it is optimal for the winner to report his shock as $\tilde{\phi}_i^*$.

2.4.1.2 The first stage

From Lemma 3, I know how the winner should manipulate the report about his realized shock if he deviates in the first stage. Therefore, I can formulate a representative buyer's problem in the first stage. Assuming all the other buyers truthfully report their initial estimates, buyer i's payoff in the first stage if he reports his initial estimate as $\tilde{w}_i$ is given by:

$$\begin{aligned} U^i(\tilde{w}_i, w_i) &= \int_{\mathcal{W}_{-i}} \int_{\oplus_i} x_i(\tilde{w}_i, w_{-i}) \tilde{U}^i(\tilde{\phi}_i^*, \phi_i; \tilde{w}_i, w_i, w_{-i}) dG_i(\phi_i) dF_{-i}(w_{-i}) \\ &\quad - \int_{\mathcal{W}_{-i}} t_i^1(\tilde{w}_i, w_{-i}) dF_{-i}(w_{-i}). \end{aligned} \tag{2.14}$$

If he wins, he gets the continuation value of the second stage, $\tilde{U}^i(\tilde{\phi}_i^*, \phi_i; \tilde{w}_i, w_i, w_{-i})$. The first round money transfer is deducted from his payoff. In the first stage, he has no information about other buyers' private information. More importantly, he does not know his shock but does know how to report his shock contingent on his realized shock.

The incentive compatibility constraints and participation constraints in the first stage require:

$$U^i(w_i, w_i) \geq U^i(\tilde{w}_i, w_i), \quad \forall i \in N, \forall \tilde{w}_i, w_i \in \mathcal{W}_i \qquad (IC1), \tag{2.15}$$

$$U^i(w_i, w_i) \geq 0, \quad \forall i \in N, \forall w_i \in \mathcal{W}_i \qquad (PC). \tag{2.16}$$

The following lemma gives the implication of the constraints.

Lemma 4 The incentive compatibility constraints and the participation constraints in the first stage, (2.15) and (2.16), are satisfied if and only if the following conditions hold:

$$\forall i \in N, \forall \tilde{w}_i, w_i \in \mathcal{W}_i,$$

$$\int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(z, \phi_i) dG_i(\phi_i) dz \geq \int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(\tilde{w}_i, z + \phi_i - \tilde{w}_i) dG_i(\phi_i) dz, , \tag{2.17}$$

$$U^i(w_i, w_i) = U^i(\underline{w}_i, \underline{w}_i) + \int_{\underline{w}_i}^{w_i} \int_{\oplus_i} Q_i(\xi, \phi_i) dG_i(\phi_i) d\xi, \quad \forall i \in N, \forall w_i \in \mathcal{W}_i, \tag{2.18}$$

$$U^i(\underline{w}_i, \underline{w}_i) \geq 0, \quad \forall i \in N. \tag{2.19}$$

The first stage incentive compatibility constraints and participation constrains require that a buyer's expected consuming probability is higher for a higher initial estimate given that he will correct his lie in the second stage. A buyer's first stage informational rent is determined by his winning probability and keeping probability, and the informational rent for the buyer with the lowest initial estimate is greater than his outside option. Equation (2.18) also implies that a buyer's expected first round money transfer is determined by his winning probability and keeping probability.

Corollary 3 The expected first round money transfer is given by:

$$\begin{aligned}
&\textstyle\int_{\mathcal{W}_{-i}} t_i^1(w)dF_{-i}(w_{-i}) \\
&= \textstyle\int_{\oplus_i}(w_i+\phi_i+C_b)Q_i(w_i,\phi_i)dG_i(\phi_i) - C_bX_i(w_i) - U^i(\underline{w}_i,\underline{w}_i) - \int_{\underline{w}_i}^{w_i}\int_{\oplus_i} Q_i(\xi,\phi_i)dG_i(\phi_i)d\xi \\
&\quad - \textstyle\int_{\mathcal{W}_{-i}}\int_{\oplus_i} x_i(w_i,w_{-i})t_i^2(w_i,w_{-i},\phi_i)dG_i(\phi_i)dF_{-i}(w_{-i}), \qquad \forall i\in N, \forall w_i\in\mathcal{W}_i,
\end{aligned} \tag{2.20}$$

where $t_i^2(w_i, w_{-i}, \phi_i)$ is given by (2.11).

2.4.1.3 The seller's problem

The seller maximizes her revenue subject to all the constraints by choosing (x, y, t^1, t^2). To summarize, the seller's problem becomes:

$$\begin{aligned}
\max_{x,y,t^1,t^2} \quad & \textstyle\sum_{i=1}^n \int_{\mathcal{W}}\int_{\oplus_i}\left[t_i^1(w_i,w_{-i}) + x_i(w_i,w_{-i})t_i^2(w_i,w_{-i},\phi_i)\right]dG_i(\phi_i)dF(w) \\
& + R\textstyle\int_{\mathcal{W}}\left[1-\sum_{i=1}^n x_i(w)\right]dF(w) - C_s\int_{\mathcal{W}}\sum_{i=1}^n x_i(w)dF(w) \\
& + S\textstyle\int_{\mathcal{W}}\int_{\oplus_i}\left\{\sum_{i=1}^n x_i(w)\left[1-y_i(w,\phi_i)\right]\right\}dG_i(\phi_i)dF(w),
\end{aligned}$$

subject to:

$$(2.2),\ (2.3),\ (2.9),\ (2.10),\ (2.17),\ (2.18),\ (2.19).$$

Her revenue is composed of four parts: money transfers from buyers, reservation value if she keeps the object and does not send the object to any buyer, and the cost of sending the object to the winner, the salvage value if the winner returns the object. Constraints (2.2) and (2.3) mean that there is only one unit of the object available. Constraints (2.9) and (2.10) are the simplified incentive compatible constraints in the second stage and constraints (2.17), (2.18) and (2.19) are the simplified incentive compatible constraints and the simplified participation constraints in the first stage. Define the following two functions:

$$J_i(w_i) = w_i - \frac{1-F_i(w_i)}{f_i(w_i)},$$

$$\mathcal{J}_j(w_j) = \int_{-J_j(w_j)+S-C_b}^{\overline{\phi}_j} [\phi_j + J_j(w_j) - S + C_b] \, dG_j(\phi_j),$$

where $J_i(w_i)$ is called the virtual initial estimate, and $\mathcal{J}_j(w_j)$ is called the modified virtual initial estimate. Another interpretation in Bulow and Roberts [9] is that a virtual valuation represents the marginal revenue to sell to a buyer. Here the modified virtual initial estimate $\mathcal{J}_i(\hat{w}_i)$ represents the marginal revenue to sell to buyer i taking into consideration that he may learn his shock later. Given that the a buyer's hazard rate function of his initial estimate is increasing, both his virtual initial estimate function and modified virtual initial estimate function are increasing in his initial estimate, i.e. $J_i'(w_i),\ \mathcal{J}_j'(w_j) \geq 0$. The following proposition characterizes the optimal two-stage direct mechanism.

Proposition 2 The mechanism (x, y, t^1, t^2) defined below represents an optimal two-stage direct mechanism.

$$x_i(w) = \begin{cases} 1 & if\ i = argmax_j \left\{ \mathcal{J}_j(w_j), R + C_s + C_b - S \right\} \\ 0 & otherwise \end{cases}, \quad \forall i \in N,\ w \in \mathcal{W}, \tag{2.21}$$

$$y_i(w, \phi_i) = \begin{cases} 1 & if\ w_i + \phi_i \geq \frac{1-F_i(w_i)}{f_i(w_i)} + S - C_b \\ 0 & otherwise \end{cases}, \qquad \forall i \in N,\ w \in \mathcal{W},\ \phi_i \in \oplus_i, \tag{2.22}$$

$$t_i^2(w, \phi_i) = \begin{cases} 0 & if\ w_i + \phi_i \geq \frac{1-F_i(w_i)}{f_i(w_i)} + S - C_b \\ -\frac{1-F_i(w_i)}{f_i(w_i)} - S & otherwise \end{cases}, \forall i \in N,\ w \in \mathcal{W},\ \phi_i \in \oplus_i, \tag{2.23}$$

$$\begin{aligned} &\int_{\mathcal{W}_{-i}} t_i^1(w) dF_{-i}(w_{-i}) \\ &= \int_{\oplus_i} (w_i + \phi_i + C_b) Q_i(w_i, \phi_i) dG_i(\phi_i) - C_b X_i(w_i) - \int_{\underline{w}_i}^{w_i} \int_{\oplus_i} Q_i(\xi, \phi_i) dG_i(\phi_i) d\xi \\ &\quad - \int_{\mathcal{W}_{-i}} \int_{\oplus_i} x_i(w_i, w_{-i}) t_i^2(w_i, w_{-i}, \phi_i) dG_i(\phi_i) dF_{-i}(w_{-i}), \qquad \forall i \in N,\ w_i \in \mathcal{W}_i. \end{aligned} \tag{2.24}$$

From equations (2.23) and (2.24), we know that the money transfer functions are fully determined by buyers' winning probabilities and keeping probabilities, which I call the allocation rules. In fact, a generalized revenue equivalency theorem holds. As long as two mechanisms induce the same allocation rules, they should generate the same expected revenue for the seller. Since the optimal two-stage direct mechanism can be fully characterized by the optimal allocations rules, it is helpful to discuss these optimal allocation rules, defined in equation (2.21) and (2.22), separately.

These optimal allocation rules are deterministic and take simple forms. The first issue is how to choose the winner optimally. According to equation (2.21), the winner should be the one with the highest modified virtual initial estimate, given that it is higher than the cutoff $\hat{w}_i$ determined by $\mathcal{J}_i(\hat{w}_i) = R + C_s + C_b - S$. When buyers' shocks are symmetric, i.e. $G_i = G, \forall\, i$, a Myerson's optimal auction with a different reserve price should be held to select the winner, since the buyer with the highest modified virtual initial estimate is the one with the highest virtual initial estimate. When buyers are **ex-ante** symmetric, i.e. $G_i = G,\ F_i = F, \forall\, i$, using a first-price or second-price auction can select the "best" buyer. The decision as to whether the winner should keep or return the object, i.e. equation (2.22), is also a cutoff rule. If the winner's realized shock is higher than the cutoff $\hat{\phi}_i$, which is equal to $-J_i(w_i) + S - C_b$, he keeps the object; otherwise he sends the object back. Note that if this cutoff is less than the winner's worst shock, i.e. $\hat{\phi}_i < \underline{\phi}_i$, the winner should always keep the object and no return should be allowed.

Since the cutoffs, $\hat{w}_i$ and $\hat{\phi}_i$, play important roles in the optimal two-stage direct mechanism, it may be helpful to do some comparative statistics to see how they are affected by the parameters, $R,\ C_s,\ C_b$, and S. Recall that $\hat{w}_i$ and $\hat{\phi}_i$ are determined by:

$$\underbrace{\int_{-J_i(\hat{w}_i)+S-C_b}^{\bar{\phi}_j} [\phi_i + J_i(\hat{w}_i) - S + C_b]\, dG_i(\phi_i)}_{\mathcal{J}_i(\hat{w}_i)} = R + C_s + C_b - S. \tag{2.25}$$

$$\hat{\phi}_i = -J_i(w_i) + S - C_b. \tag{2.26}$$

Corollary 4 $\frac{\partial \hat{w}_i}{\partial R} \geq 0$, $\frac{\partial \hat{w}_i}{\partial C_s} \geq 0$, $\frac{\partial \hat{w}_i}{\partial C_b} \geq 0$, $\frac{\partial \hat{w}_i}{\partial S} \leq 0$, $\frac{\partial \hat{\phi}_i}{\partial R} = 0$, $\frac{\partial \hat{\phi}_i}{\partial C_s} = 0$, $\frac{\partial \hat{\phi}_i}{\partial C_b} < 0$, $\frac{\partial \hat{\phi}_i}{\partial S} > 0$.

When the seller's reservation value is higher, she keeps the object for herself more often as in Myerson's optimal auction. It is also intuitive that when the cost of sending the object to the winner increases, the seller keeps the object more often. When the cost for the winner to send back the object increases, the seller also keeps the object more often. This is because a buyer values the object less due to a bigger loss when he has to return the object, and therefore, the seller can make less profit by selling the object. When the salvage value increases, the seller is more willing to send out the object. Even if the winner returns the object, she does not lose too much. Suppose $C_b = C_s = 0$ and $R = S$, the seller should always send out the object to a buyer. This is because by sending out the object she can get at least R which is the same as keeping the object.

Both the seller's reservation value and the cost for the seller to send the object have no effect on the seller's decision as to whether the winner should return the object. This is intuitive. As long as the seller decides to send the object, the reservation value is irrelevant and the cost of sending the object is already sunk. In contrast, the higher the cost for the winner to send back the object is, the less frequent the returns. Furthermore, when the salvage value of the object gets smaller, fewer returns will occur in the optimal mechanism. Either decreasing the salvage value or increasing the cost for the winner to send back the object can lead to the situation in which no return should be allowed under any circumstance. This gives a potential explanation why in some auctions, such as house auctions, flower auctions, and wine auctions, sellers do not allow any return. The salvage values in flower auctions and wine auctions are too low and the transaction costs to return in house auctions are too high.

I have discussed the optimal allocation rules above. In addition, $\{y_i(w, \phi_i), t_i^2(w, \phi_i))\}_{i=1}^n$ is worthy of discussion in its own right, since it characterizes the optimal refund policy. Note that the optimal refund policy depends only on the winner's initial estimate and his shock, not on the number of buyers and the losers' types. This means that the optimal two-stage direct mechanism is separable. Competition among buyers only affect the selection of the winner. The seller should provide the same refund policy for the winner as if he were the only buyer. In this optimal refund policy, when the winner's realized shock is higher than the cutoff, he keeps the object and no further money transfer is made; when his realized shock is lower than the cutoff, he should return the object for a refund. Several important features are summarized in the following Corollaries.

Corollary 5 In the optimal refund policy, there is excessive return for the winner unless his initial estimate is at its upper bound. The distortion is smaller if the winner has a higher initial estimate.

This means the winner returns the object too often compared with the efficient one, where the winner keeps the object if his **ex-post** valuation is higher than $S - C_b$ and returns it if otherwise. The distortion is introduced only by the fact that initial estimates are buyers' private information. The fact that the winner may privately learn of shock does not introduce any distortion. The hazard rate measures the distortion introduced by eliciting truthful information. Since I assume that the hazard rate is increasing, the distortion is small for higher initial estimate. To induce this allocation, there are some requirements on the amount of the refund.

Corollary 6 In the optimal refund policy, the amount of the refund the winner could get is decreasing in his initial estimate.

This suggests that the commonly observed linear refund policy is actually never optimal. In the second-price auction, the amount of the refund depends only on the second highest bid and not on the winner's initial estimate. In the first-price auction, the amount of the refund is an increasing function of the winner's bid and, therefore, is increasing in the winner's initial estimate.

Finally, the distribution of the shocks does not affect the optimal refund policy. But it affects the selection of the winner. If a buyer's shock is uniformly distributed, the selection rule favors him if his shock has a larger variance. This is actually intuitive since there is more room to improve the efficiency if the shock is more variate.

2.4.1.4 Implementing the optimal two-stage direct mechanism by a first-price or second-price auction with a refund policy

The following is an example in which a standard auction with a refund policy can actually implements the optimal two-stage direct mechanism. I assume that all buyers are **ex-ante** identical, i.e. $F_i = F$ and $G_i = G$.

Since all buyers are **ex-ante** symmetric, the one with the highest modified virtual initial estimate is the one with the highest initial estimate. Intuitively, a standard auction should be able to select the best buyer. As shown in the following proposition, this is indeed the case and the optimal two-stage direct mechanism can be implemented by a first-price or second-price auction with a refund policy.

Proposition 3 A second-price auction with a reserve price $B_2(\mathcal{J}^{-1}(R + C_s - S + C_b))$ and a fee $C_2(p, q) = p - \frac{1-F(B_2^{-1}(q))}{f(B_2^{-1}(q))} - S$ implements the optimal two-stage direct mechanism, where p is the transaction price in the auction, q is the winner's bid in the auction, and

$$\begin{aligned} B_2(w) \quad &= \{\int_{-w+\frac{1-F(w)}{f(w)}+S-C_b}^{\overline{\phi}} [w + \phi_i - \tfrac{1-F(w)}{f(w)} - S + C_b] dG(\phi_i) \\ &\qquad + [\tfrac{1-F(w)}{f(w)} + S - C_b]\} \\ &\quad + \tfrac{F(w)}{(n-1)f(w)} G(-w + \tfrac{1-F(w)}{f(w)} + S - C_b) \frac{d[\frac{1-F(w)}{f(w)}]}{dw} \end{aligned} \tag{2.27}$$

Since the fee needs to be constructed in a way such that the winner, buyer i, with a shock higher than the cutoff $\hat{\phi}_i$ keeps the object, it depends on both the transaction price in the auction as well as the winner's bid in the auction, which conveys accurate information about the winner's initial estimate in equilibrium. This also suggests that an English auction with a refund policy would generally not be able to implement the optimal two-stage direct mechanism, since the seller has no information about the winner's bid and she cannot make the fee contingent on the winner's initial estimate.

Proposition 4 A first-price auction with a reserve price $\mathcal{J}^{-1}(R + C_s - S + C_b)$ and a fee $C_1(p) = p - \frac{1-F(B_1^{-1}(p))}{f(B_1^{-1}(p))} - S$ implements the optimal two-stage direct mechanism, where p is the transaction price in the auction, and

$$\begin{aligned} B_1(w) \quad &= \tfrac{1}{F(w)^{n-1}} \int_{\mathcal{J}^{-1}(R+C_s-S+C_b)}^{w} \left\{ \mathcal{J}(\xi) + \left[\tfrac{1-F(\xi)}{f(\xi)} + S - C_b \right] \right\} dF(\xi)^{n-1} \\ &\quad + \tfrac{1}{F(w)^{n-1}} \int_{\mathcal{J}^{-1}(R+C_s+S+C_b)}^{w} F(\xi)^{n-1} G(-\xi + \tfrac{1-F(\xi)}{f(\xi)} + S - C_b) d \left[\tfrac{1-F(\xi)}{f(\xi)} \right] \\ &\quad + \tfrac{F(\mathcal{J}^{-1}(R+C_s-S+C_b))^{n-1}}{F(w)^{n-1}} \mathcal{J}^{-1}(R + C_s - S + C_b). \end{aligned} \tag{2.28}$$

In contrast to the second-price auction above, the fee only depends on the transaction price. This is because the transaction price also conveys accurate information about the winner's initial estimate in equilibrium.

In both of the auctions above, when the winner decides to return the object, he gets a refund, $\frac{1-F(B^{-1}(p))}{f(B^{-1}(p))}+S$; when the winner decides to keep the object, no more money transfer is made. Since $B_1(w)$ and $B_2(w)$ are constructed in such ways that they are the bidding functions in the auction stage, the refund becomes $-\frac{1-F(w)}{f(w)} - S$. Therefore, the second round money transfer is consistent to that in equation (2.23). In addition, a buyer's situations in the above auctions are different from the situations in standard ones. In contrast to standard auctions, the value of winning the object in the above auctions is endogenously given and it depends on both of one's initial estimate and its report. A buyer's report also acts as a signal of his initial estimate, since how much the winner should pay to return the object depends on his reported initial estimate. When there are signaling effects in an auction, the existence of an equilibrium is always a concern. However, as shown in the proofs, in the above auctions, the single crossing property is always satisfied, and this ensures the existence of an equilibrium. Finally, in standard second-price and first-price auctions, the optimal reserve prices are the same; in contrast, the optimal reserve prices in the above two auctions are not equal.

2.4.2 The relaxed environment

I restrict attention to a class of two-stage direct mechanisms and find the optimal one among this class above. Now I need to show that this optimal two-stage direct mechanism is optimal in the relaxed environment to conclude that it is optimal in the original environment. In the relaxed environment, I assume that the seller can actually observe the winner's shock when it is realized. In this relaxed environment, the revelation principle applies. Any feasible mechanism can be represented by a direct mechanism. Since only buyers' initial estimates are private information, direct mechanisms are easy to characterize.

In a direct mechanism, buyers are asked to report their initial estimates $w = (\tilde{w}_1, \cdots, \tilde{w}_n)$. The seller sends the object to buyer i with probability $x_i(\tilde{w})$ and asks him to keep the object with probability $y_i(\tilde{w}, \phi_i)$ if his realized shock is ϕ_i. Contingent money transfers $t_{ij}(\tilde{w}, \phi_i)$ are made from buyer j to the seller if buyer i is the winner. Let $x(\tilde{w}) = (x_1(\tilde{w}), \cdots, x_n(\tilde{w}))$ and $t = (t_{11}(\tilde{w}), \cdots, t_{1n}(\tilde{w}), \cdots, t_{nn}(\tilde{w}))$.

Given all other buyers truthfully report their initial estimates, buyer i's expected payoff if he reports his initial estimate as $\tilde{w}_i$ is:

$$
\begin{aligned}
U^i(\tilde{w}_i, w_i) \quad &= \textstyle\int_{\mathcal{W}_{-i}} \int_{\oplus_i} (w_i + \phi_i) x_i(\tilde{w}_i, w_{-i}) y_i(\tilde{w}_i, w_{-i}, \phi_i) dG_i(\phi_i) dF_{-i}(w_{-i}) \\
&\quad - C_b \textstyle\int_{\mathcal{W}_{-i}} \int_{\oplus_i} x_i(\tilde{w}_i, w_{-i}) \left[1 - y_i(\tilde{w}_i, w_{-i}, \phi_i)\right] dG_i(\phi_i) dF_{-i}(w_{-i}) \\
&\quad - \textstyle\sum_{j=1}^n \int_{\mathcal{W}_{-i}} \int_{\oplus_j} x_j(\tilde{w}_i, w_{-i}) t_{ji}^2(\tilde{w}_i, w_{-i}, \phi_j) dG_j(\phi_j) dF_{-j}(w_{-j}) \qquad (2.29)
\end{aligned}
$$

$$
= \textstyle\int_{\oplus_i} (w_i + \phi_i + C_b) Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i) - C_b X_i(\tilde{w}_i) - \hat{T}_i(\tilde{w}_i),
$$

where $\hat{T}_i(\tilde{w}_i) = \sum_{j=1}^n \int_{\mathcal{W}_{-i}} \int_{\oplus_j} x_j(\tilde{w}_i, w_{-i}) t_{ji}^2(\tilde{w}_i, w_{-i}, \phi_j) dG_j(\phi_j) dF_{-j}(w_{-j})$.

If he wins and then keeps the object, he gets the utility of consuming it; if he wins but has to return the object, he incurs the cost C_b to send it back to the seller; money transfers are deducted from his payoff function.

Now, the seller's problem is to maximize her expected revenue subject to all the constraints by choosing (x, y, t). Formally,

$$
\begin{aligned}
\max_{x,y,t} \quad & \textstyle\sum_{i=1}^n \int_{\mathcal{W}_i} \hat{T}_i(w_i) dF_i(w_i) + R \int_{\mathcal{W}} \left[1 - \sum_{i=1}^n x_i(w)\right] dF(w) \\
& + S \textstyle\int_{\mathcal{W}} \int_{\oplus_i} \left\{\sum_{i=1}^n x_i(w) \left[1 - y_i(w, \phi_i)\right]\right\} dG_i(\phi_i) dF(w) - C_s \int_{\mathcal{W}} \sum_{i=1}^n x_i(w) dF(w),
\end{aligned}
$$

$$U^i(w_i, w_i) \geq U^i(\tilde{w}_i, w_i), \quad \forall i \in N, \forall \tilde{w}_i, w_i \in \mathcal{W}_i \qquad (IC), \tag{2.30}$$

$$U^i(w_i, w_i) \geq 0, \quad \forall i \in N, \forall w \in \mathcal{W} \qquad (PC), \tag{2.31}$$

$$\sum_{i=1}^{n} x_i(w) \leq 1 \; and \; x_i(w) \geq 0, \qquad \forall i \in N, \forall w \in \mathcal{W}, \tag{2.32}$$

$$y_i(w, \phi_i) \leq 1 \; and \; y_i(w, \phi_i) \geq 0, \qquad \forall i \in N, \; \forall w \in \mathcal{W}, \; \forall \phi_i \in \oplus_i, \tag{2.33}$$

Constraints (2.30) are the incentive compatibility constraints. Constraints (2.31) are the participation constraints. Constraints (2.32) and (2.33) mean that there is only one unit of the object available. The set of (x, y, t) that satisfies all the constraints is called the set of feasible mechanisms. The following lemma simplifies the characterization of the feasible mechanism.

Lemma 5 (x, y, t) is feasible if and only if the following conditions hold:

$$if \; w_i \geq \tilde{w}_i, \; then \int_{\oplus_i} Q_i(w_i, \phi_i) dG_i(\phi_i) \geq \int_{\oplus_i} Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i), \quad \forall i \in N, \forall \tilde{w}_i, w_i \in \mathcal{W}_i, \tag{2.34}$$

$$U^i(w_i, w_i) = U^i(\underline{w}_i, \underline{w}_i) + \int_{\underline{w}_i}^{w_i} \int_{\oplus_i} Q_i(\xi, \phi_i) dG_i(\phi_i) d\xi, \quad \forall i \in N, \forall w \in \mathcal{W}, \tag{2.35}$$

$$U_i^i(\underline{w}_i, \underline{w}_i) \geq 0, \quad \forall i \in N, \tag{2.36}$$

$$(2.32) \; and \; (2.33).$$

Inequality (2.34) requires that a buyer's expected consuming probability is increasing. Equation (2.35) implies that a buyer's informational rent, or equivalently his expected money transfer, is determined by his winning probability and keeping probability. It turns out that the allocation rules in the optimal two-stage direct mechanism are also optimal in the relaxed environment.

Proposition 5 When the winner's realized shock can be observed by the seller, the optimal selling mechanism generates the same revenue as that in Proposition 2.

The crucial observation is that equation (2.35) is the same as equation (2.18). This implies that buyers' informational rents at the beginning of the contract are the same in both the relaxed environment and the two-stage direct mechanism given that the allocation rules are the same. It turns out that if I substitute this equation into the objective function of the seller, it coincides with the objective function in the two-stage direct mechanisms. Therefore, the only difference between the seller's problem in the two-stage direct mechanism and in the relaxed environment is that (2.9) and (2.17) are now replaced by (2.34). It is not hard to see that (2.34) is less restrictive than (2.9) and (2.17). Also, recall that the way I found the optimal two-stage direct mechanism is to first ignore equation (2.9) and (2.17) and then prove that the optimal two-stage direct mechanism satisfies those two constraints. Therefore, the optimal two-stage direct mechanism should also be optimal in the relaxed environment.

This means that buyers cannot gain any informational rents for their shocks that they may privately learn later. The intuition is as follows. At the very beginning, buyers have no information advantages over the seller regarding their shocks. Since the seller is proposing the contract and has all the bargaining power, she extracts all the benefit.

2.5 Conclusion

In this chapter, I investigate refund policies in auctions. Providing a refund policy induces buyers to bid more aggressively. The more generous the refund policy is, the more aggressively buyers bid. If the refund policy is too strict, it is actually equivalent to not allowing for returns. I also find the optimal selling mechanism in the abstract environment. In this optimal selling mechanism, the seller chooses to sell to the buyer with the highest modified virtual initial estimate if it is higher than a cutoff, and allows him to return the object if his realized shock is lower than a cutoff. An auction with a refund policy can represent the optimal selling mechanism. This mechanism is separable and the optimal refund policy takes a simple and conventional form. This optimal selling mechanism can be implemented by a first-price or second-price auction with a refund policy when buyers are **ex-ante** symmetric. Finally, even if the seller cannot observe the winner's realized shock, she can actually achieve the same revenue as if she could. This means that buyers gain informational rent only for the existing private information; any private information realized after the contract is signed cannot generate informational rent for them.

In this chapter, I assume that W_i and Φ_i are independent to each other and across buyers. It is natural to ask what will change if some interdependencies are introduced in the model.[10] First, let us look at the case that buyers' shocks are correlated. One good example would be the wine auctions. If the wine is corked, then everybody should treat this as a negative shock.[11] In most wine auctions, wine can be returned for a refund if it is corked. However, this modification does not change the results too much. I only need to redefine the G_i as the marginal distribution of buyer i's shock and all the results still hold. This is because only the winner can observe the shock.

Second, let us consider the case that a buyer's initial estimate and his shock are correlated. This will bring some difficulties to solve the model, since the observation that the privately observed shocks do not generate any information rents for buyers does not hold anymore. Therefore, we can not use the procedure above to solve the model. Consider an extreme case that the initial estimate and the shock are perfectly correlated, then there is no need to provide a refund policy. This is because there is no more uncertainty and a Myerson's optimal auction based on buyers' initial estimates is optimal. I conjecture that the seller would be less willing to provide a refund policy if there is stronger correlation between a buyer's initial estimate and

[10] I maintain the assumption that the seller and buyers share the same information at the beginning.
[11] At the beginning, the seller usually does not know whether the wine is corked or not either.

his shock.

CHAPTER 3

THE ROLE OF INFORMATION REVELATION IN ELIMINATION CONTESTS

3.1 Related literature

My essay is an attempt to respond to Moldovanu and Sela [50], who write:“ while my model is such that the information released in the dynamic contest has a very simple structure, I believe that an interesting avenue is to focus on the role of information in contests with multiple round.” Lai and Matros [38] have already considered a two-round elimination contest with full revelation or no revelation of bidders’ bids and characterized the corresponding equilibria. Their model is more general than ours, allowing for more players dividing into more groups in the first round. Furthermore, there are interim prizes for the first round winners. But their approach is different from ours. They assume that the first-round winners are committed to act according to their pretended type in the second round in a deviation. Furthermore, a deviator’s valuation also becomes that of the pretended type. That is, when a player deviates by pretending to be a different type, he becomes that pretended type in the second round. Here, I assume that when a player deviates, he maintains his original type.

The full information revelation is related to the literature on signaling in auctions, which also find the non-existence of efficient equilibrium. Goeree [21] considers an auction with an aftermarket and bidders compete for the advantage in the strategic interactions in the aftermarket. The winning bid in the auction is revealed. Haile [26] [27] [28] considers a model of an auction followed by a resale auction offered by the winner of the first auction. The bidders have some noisy private signals regarding their valuations coming into the first auction and their bids are revealed before the resale auction. In these papers, efficient symmetric separating Perfect Bayesian Nash Equilibria are investigated and it is found that such separating equilibria may not exist. Goeree [21] states that "if bidders want to understate their private information, a separating equilibrium may fail to exist when the incentives to signal via a lower bid are stronger for higher valuations". Furthermore, Haile [28] finds that, due to signaling effects, player's objective function is not quasiconcave, which contributes to the failure of the existence of a separating equilibrium. In yet another recent paper, Cai et al [10] investigate sequential first-price and second-price auctions of multiple units. They find that when all bids are revealed after each auction, there exists no equilibrium with a symmetric strictly increasing bidding function. Similarly to my model, bidders have different incentives to increase and to decrease their bids, and no strictly monotone bidding function can satisfy the first order conditions implied by the incentive compatibility constraints. Therefore, bidders with the highest valuations may not win the objects.

These results in the literature show that the non-existence of efficient equilibria is not unique for elimination contests. When important information is released in the interim stages of a game, it alters the incentives of the players in the earlier stages. In an environment where the later rounds winning probability is not continuous in one's bid (and thus type), the incentives to influence other players' beliefs of one's type upward and downward are different. As a result, equilibria with strictly monotone behavior do not exist.

My partial revelation rule is related to the literature on sequential auctions.[1] In that literature, multiple units of identical objects are auctioned off one by one. Each bidder demands at most one unit of the object. After each auction, only the winning bid is revealed (except in the above-mentioned paper by Cai et al [10]). This informational structure avoids the complicated effects of signaling by bids, since the only potential bidder who could signal with his bid, the winner, would not be in later auctions to benefit from the signaling. Likewise, in this essay, I assume that only bids from those bidders being eliminated are revealed. Thus those bidders who could manipulate their signals are excluded from the next round competition, and this simplifies the analysis.

The no revelation rule has already been investigated by Moldovanu and Sela [50]. In their model, the first round winners know only who has won and who has lost. They characterize the unique symmetric Perfect Bayesian Nash Equilibrium in which players use strictly increasing bidding functions in both rounds of the contest. Therefore, the no revelation rule leads to an efficient allocation of the prize in the sense that the player with the highest valuation always wins the prize.

3.2 The Model

I consider a two-round elimination contest. There are 4 risk neutral players in the contest for some prize. They have private information regarding their valuations of the prize. Player i's valuation is V_i, and its realization is denoted by v_i. These V_i's are drawn independently from an identical distribution $F(\cdot)$, with density function $f(\cdot)$ and support $\mathcal{V} = [\underline{v}, \overline{v}]$. I assume that both the valuation and its density are bounded away from 0; i.e. $0 < \underline{v} < \overline{v} < +\infty$ and

[1] See McAfee and Vincent [47], Mezzetti et al [49], and Weber [63]

$0 < f(v) < +\infty$ for $v \in [\underline{v}, \overline{v}]$.

Each contest is in the form of an all-pay auction; the player with the highest bid (effort) wins but every player pays his own bid. If there is a tie, I assume that the stronger player wins. This is a standard assumption; it is made to avoid the problem that the stronger player may want to outbid his rival by an infinitesimal amount. If two players are equally strong, then each player wins with equal probability. Different situations require different definitions of a stronger player and I provide the precise definition when it is needed. The timing of the game is as follows.

(1) 4 players are evenly divided into 2 groups: group A (players 1 and 2) and group B (players 3 and 4).

(2) Nature draws a valuation for each player from the same distribution $F(\cdot)$ independently.

(3) In each group, a first (preliminary) round contest is held. The winners from each group, denoted by player A and player B, enter the second (final) round; the losers, denoted by player a and player b, are eliminated.

(4) Certain information (to be specified later) is revealed after the first round and before the second round.

(5) Players A and B compete in the second round.

The rules governing the information revelation between rounds (i.e. stage 4) are very important in the analysis. I investigate the following three information revelation rules: full revelation, partial revelation and no revelation.

Before offering more details of the model, I first discuss the equilibrium I will be looking for throughout the chapter: symmetric separating Perfect Bayesian Nash Equilibrium (PBNE). “ Symmetric separating ” means that, in the first round, all players adopt the same strictly increasing bidding function. A PBNE consists of strategy profiles and a system of beliefs for the players. The strategy profile is sequentially rational for each player given the beliefs and the beliefs satisfy the Bayes’ rule whenever possible. Since all the players are **ex-ante** symmetric, it is natural to look at a symmetric equilibrium. I focus on a symmetric separating equilibrium since it is the only equilibrium that the player with the highest valuation always wins and thus the outcome is always efficient.

I analyze my model by constructing the equilibrium strategies. I first assume that, in the first round, all players adopt the same strictly increasing bidding function. I then prove that such a bidding function does or does not exist in equilibrium. More specifically, after making the assumption above, I use backward induction to analyze this two-round game. I start with characterizing the equilibrium in the second-round continuation game when no one deviated in the first round. I then characterize the equilibrium in the second-round continuation game when only one player deviated in the first round. Note that the deviation matters in the analysis only when this deviating player gets into the second round. Finally, I characterize the strictly increasing bidding function in the first round if it exists or show that there exists no such function.

In the following three sections, I analyze three different information revelation rules in sequence.

3.3 Full Revelation

I first analyze the full revelation rule. All bids in the first round contest are revealed. This is the most interesting case as I need to deal with signaling effects in the analysis. When a player deviates from his equilibrium strategy and pretends to have a different valuation, he is mistaken by the other player in the second round as that pretended type, and thus receives responds aimed at that type. In this way, a player can manipulate the strategies of the other player by choosing different pretended valuations. Of course, in equilibrium, each player will follow his equilibrium strategy.

Assume that the equilibrium bidding function in the first round is strictly increasing. I first examine the second round interactions.

3.3.1 Second Round Strategies

Suppose that in the first round all players adopt the same strictly increasing bidding function $b^P(\cdot)$. I denote the image of the first round strategy as $b^P(\mathcal{V}) = [b^P(\underline{v}), b^P(\overline{v})]$. It is not hard to see that the lower bound of $b^P(\mathcal{V})$ is $b^P(\underline{v}) = 0$, since the player with the lowest valuation has zero probability of entering the second round and thus would not bid more than zero. If a player bids $b \in b^P(\mathcal{V})$ in the first round, then according to Bayes' rule, other players in the second round believe that his valuation is $(b^P)^{-1}(b)$. If a player's bid is outside the image of the first round bidding function, i.e. $b > b^P(\overline{v})$,[2] I assume that all other players believe that he has valuation $\overline{v}$. As a result, bidding $b^P(\overline{v})$ strictly dominates bidding $b > b^P(\overline{v})$, since there is no benefit for the higher bid while the cost is higher. Under this specification of the off-path beliefs, no player has any incentive to deviate to a valuation outside the support of the valuation

[2] Since players can not bid a negative amount, the only possible bid which is outside the image of the first round bidding function is $b > b^P(\overline{v})$.

space $\mathcal{V}$. Therefore, I will focus on deviations of bids within the image of the first round bidding function.

In the second round, the finalists update their beliefs on each other's valuations using Bayes' rule by inverting the bidding function. Hence, in the non-deviated continuation game where all players follow the first round equilibrium bidding function, the second round contest becomes an all pay auction with complete information. This game has been extensively studied and I reproduce below the following lemma in Baye et al [7].

Lemma 6 Suppose that two players i and j with $v_j \leq v_i$ compete in an all-pay auction for a unique prize with complete information. In the unique Nash equilibrium, both players randomize on the interval $[0, v_j]$. Player i's bid is uniformly distributed according to $c.d.f.\ G_i(b_i) = \frac{b_i}{v_j}$. Player j's bid is distributed according to the $c.d.f.\ G_j(b_j) = \frac{v_i - v_j + b_j}{v_i}$, with a mass point at 0. The winning probabilities are respectively $q_i = 1 - \frac{v_j}{2v_i}$ and $q_j = \frac{v_j}{2v_i}$. The expected payoffs are respectively $\Pi_i = v_i - v_j$ and $\Pi_j = 0$.

In order to characterize the equilibrium bidding function in the first round, I need to examine one special deviated continuation game in the second round. In this continuation game, only one player deviates and pretends to have a different valuation in the first round and wins the right to enter the second round. Since all players are **ex-ante** symmetric, I choose player 1 as the representative player and assume that he is the one who deviates. Let $w = (b^P)^{-1}(b) \in \mathcal{V}$ be the valuation other players believe player 1 has; it may or may not be player 1's true valuation. If he loses in the first round, then it does not affect the second round contest. The following analysis applies when he wins and enters the second round, and thus becomes player A.

In the second round, player B, the winner from group B, infers that player A's valuation is w. Player A learns that player B's valuation is v_B from his first round bid. Furthermore, player A knows that player B believes that he has valuation w. Of course, player A knows that his own valuation is actually v_A. The following lemma describes the finalists' strategies in this special deviated continuation game. Here, I designate the player with the higher bid (and thus the implied valuation) in the first round as the strong player.

Lemma 7 In the special deviated continuation game in the second round described above, player B randomizes his bid according to the following $c.d.f.$ function:

$$G_B(b_B) = \begin{cases} \frac{w-v_B+b_B}{w}, & \text{if } w \geq v_B; \\ \frac{b_B}{w}, & \text{if } w \leq v_B. \end{cases} \tag{3.1}$$

Player A bids

$$b_A^F(w, v_A, v_B) = \begin{cases} 0, & \text{if } w > v_A \text{ and } w > v_B; \\ 0, & \text{if } w > v_A \text{ and } w \leq v_B; \\ v_B, & \text{if } w \leq v_A \text{ and } w > v_B; \\ w, & \text{if } w \leq v_A \text{ and } w \leq v_B. \end{cases} \tag{3.2}$$

Player A's payoff is

$$\Pi_A^F(w, v_A, v_B) = \begin{cases} \frac{v_A(w-v_B)}{w}, & \text{if } w > v_A \text{ and } w > v_B; \\ 0, & \text{if } w > v_A \text{ and } w \leq v_B; \\ v_A - v_B, & \text{if } w \leq v_A \text{ and } w > v_B; \\ v_A - w, & \text{if } w \leq v_A \text{ and } w \leq v_B. \end{cases}$$

3.3.2 First Round Strategy

I now consider the first round bidding function. Lemma 7 gives players' strategies in the special deviated continuation game when player 1 deviates and enters the second round. Obviously, player 1' expected payoff from entering the second round depends on both his true valuation and his pretended valuation. Suppose that in the first round all other players adopt the equilibrium bidding function $b^P(\cdot)$ and player 1 has valuation v_1 but pretends to have valuation w. Then player 1' payoff in the whole game is given by

$$\begin{aligned}
& \Pi_1^P(w, v_1) \\
= \;& E\{\Pi_1^F(w, v_1, V_B)\mathcal{I}_{\{w>V_2\}}|v_1\} - b^P(w) \\
= \;& E\{\mathcal{I}_{\{w>V_2\}}\}E\{\Pi_1^F(w, v_1, V_B)|v_1\} - b^P(w) \\
= \;& F(w)\Big(E\{\Pi_1^F(w, v_1, V_B)\mathcal{I}_{\{w>V_A, w>V_B\}}|v_1\} + E\{\Pi_1^F(w, v_1, V_B)\mathcal{I}_{\{w>V_A, w\leq V_B\}}|v_1\} \\
& +E\{\Pi_1^F(w, v_1, V_B)\mathcal{I}_{\{w\leq V_A, w>V_B\}}|v_1\} + E\{\Pi_1^F(w, v_1, V_B)\mathcal{I}_{\{w\leq V_A, w\leq V_B\}}|v_1\}\Big) \\
& -b^P(w),
\end{aligned}$$

where all expectations are taken on the upper case variables in the above equation, and for the rest of the Chapter as well. Unfortunately, as I show in the following proposition, there exists no symmetric separating equilibrium in this game.

Proposition 6 In the elimination contest with full revelation rule, there does not exist any symmetric separating equilibrium in which all players adopt the same strictly increasing bidding function in the first round.

The intuition behind this proposition is as follows. In a separating equilibrium with full revelation, the players in the second round have complete information regarding each other's valuation. However, the expected payoff in the whole game calculated at the start of the first round contest is not differentiable in the player's bid at the value of his equilibrium bid. This is because in order to ensure that a player does not want to pretend to have a lower valuation, the bidding function in the first round needs to be relatively flat; conversely, in order to ensure that a player does not want to pretend to have a higher valuation, that bidding function needs to be relatively steep. There is no bidding function that can satisfy these incentive compatibility constraints in both directions.

To understand this, consider a regular one-round auction. An increasing in one's bid has two effects. First, it increases the expected winning probability. Second, it increases the expected payment. In equilibrium, the bidding function is constructed in such way that these two effects cancel out at his true valuation. In my two-round elimination contest model, in additional to the two effects above, there is a third effect – it also changes the valuation conditional on winning. However, this valuation function is not a smooth function. Upon a player's overbidding, the valuation is increasing in his pretended type. Upon a player's underbidding, the valuation is decreasing in his pretended type. To cancel this effect, there has to be a kink in the bidding function at his true valuation. Since this happens at every valuation, kinks have to be everywhere in the bidding function. Obviously, no such bidding function can exist. Therefore, if an equilibrium exists, it must involves either non-monotone bids, pooling of bids, or mixed strategies in the first round. This implies that the entry to the second round contest is not efficient.

3.3.3 Robustness of the Non-Existence Result

The main driving force for the non-existence of a symmetric separating equilibrium is the kinks in a player's first round payoff function. Do these kinks disappear if I make some small changes to the model? Here, I discuss two modifications to the model.

The first modification is to add interim prizes for the first round winners. A player who wins the first round will win a prize in addition to entering the second round contest. This does not smooth the kinks, however. This is because a player's payoff now has two parts. One is the expected value of entering the second round to win the final prize. The other one is this interim prize. The first part behaves exactly the same as what I have analyzed in the previous subsection. The second part is a smooth component. Adding a smooth component to a kinked function would not smooth the kinks, and there still exists no symmetric separating equilibrium.

The second modification is to introduce some shocks on the players' valuations after the first round. Suppose that at the beginning the game, player i's valuation is v_i. However, after he wins the first round, his valuation is subject to a random shock ϕ_i with mean zero and distribution $H(\phi_i)$ on $[\underline{\phi}, \overline{\phi}]$. I assume that the prize is always desirable, which means even with the worst shock one's valuation on the prize is still positive. The shocks are independently and identically distributed across all players. Assume that the realizations of these shocks are common knowledge.

I follow the same procedure as in the above subsections to solve the model. Given the realization of the shocks ϕ_i and ϕ_j, I can apply Lemma 6 and obtain the winners' equilibrium strategies in the non-deviated continuation game in the second round. Note that now their valuations are $v_i + \phi_i$ and $v_j + \phi_j$. In the special deviated continuation game in the second round, I get a parallel Lemma to Lemma 7. The proof is very similar, and therefore omitted here.

Lemma 8 In the special deviated continuation game in the second round described above, player B randomizes his bid according to the following cdf function:

$$G_B(b_B) = \begin{cases} \frac{w+\phi_A - v_B - \phi_B + b_B}{w+\phi_A}, & \text{if } w + \phi_A \geq v_B + \phi_B; \\ \frac{b_B}{w+\phi_A}, & \text{if } w + \phi_A \leq v_B + \phi_B. \end{cases} \tag{3.3}$$

Player A bids

$$b_A^F(w, v_A, v_B, \phi_A, \phi_B) = \begin{cases} 0, & \text{if } w > v_A \text{ and } w + \phi_A > v_B + \phi_B; \\ 0, & \text{if } w > v_A \text{ and } w + \phi_A \le v_B + \phi_B; \\ v_B + \phi_B, & \text{if } w \le v_A \text{ and } w + \phi_A > v_B + \phi_B; \\ w + \phi_A, & \text{if } w \le v_A \text{ and } w + \phi_A \le v_B + \phi_B. \end{cases} \tag{3.4}$$

Player A's payoff is

$$\Pi_A^F(w, v_A, v_B) = \begin{cases} \frac{(v_A+\phi_A)(w+\phi_A-v_B-\phi_B)}{w+\phi_A}, & \text{if } w > v_A \text{ and } w + \phi_A > v_B + \phi_B; \\ 0, & \text{if } w > v_A \text{ and } w + \phi_A \le v_B + \phi_B; \\ v_A + \phi_A - v_B - \phi_B, & \text{if } w \le v_A \text{ and } w + \phi_A > v_B + \phi_B; \\ v_A - w, & \text{if } w \le v_A \text{ and } w + \phi_A \le v_B + \phi_B. \end{cases} \tag{3.5}$$

Given the above lemma, I can formulate a player's problem at the beginning of the first round and determine the conditions under which a separating equilibrium must satisfy. Again, I obtain the non-existence result.

Proposition 7 In the elimination contest with full revelation rule and publicly observed interim valuation shocks, there does not exist any symmetric separating equilibrium in which all players adopt the same strictly increasing bidding function in the first round.

Note that when the random shocks are equal to zero, the analysis is exactly the same as in Proposition 6. The intuition for these random shocks not being able to smooth the kinks is similar to the intuition in the first modification. For each realization of the random shock, there is a kink in the payoff function. Taking the expectation at the beginning of the first round does not iron out these kinks.

Making the above random shocks the private information of the players would not smooth out the kinks either. The reason for the kink is because of the discontinuity in the deviated first round winner's best response function in the special deviated continuation game in the second round. This feature remains the same here. Suppose finalist i has under-reported his type in the first round, then it is optimal to bid zero in the second round when the shock is sufficiently low. If he over reports his type in the first round, then it is optimal to bid the upper bound of the bidding function in the second round when the shock is sufficiently high. Both situations occur with strictly positive probabilities and it makes the best response function discontinuous in his reported type in the first round. Therefore, kinks remain in the payoff function.

3.4 Partial Revelation

In this section, I analyze the case where the designer can conceal the efforts (bids) of the winners, but not those of the losers. Therefore, after the first round of contests, the bids of the eliminated players are revealed, and the bids of the winners remain concealed. Below, I analyze how this change in information revelation affects players' behavior and whether a symmetric separating equilibrium exists.

3.4.1 Second Round Strategy

I first examine a non-deviated continuation game in the second round; all players follow their equilibrium strategy in the first round. According to the rule, the bids of the eliminated players are revealed. After seeing these bids, the finalists (winners), player A and player B, update their beliefs regarding each other's valuation. Since the bidding function in the first round is strictly increasing, they can calculate the losers' valuations v_a and v_b by inverting the bidding function, and thus v_a and v_b are fully revealed. Meanwhile, it is common knowledge that players A and B are the winners in their own groups. Using Bayes' rule, player A believes that player B's valuation is distributed on $[v_b, \overline{v}]$ with $c.d.f.$ $\frac{F(v_B)-F(v_b)}{1-F(v_b)}$; and player B believes that player A's valuation is distributed on $[v_a, \overline{v}]$ with $c.d.f.$ $\frac{F(v_A)-F(v_a)}{1-F(v_a)}$. Since $prob(v_a = v_b) = 0$, players are in an asymmetric all pay auction with probability one. There are many papers investigating asymmetric auctions.[3] It is impossible to obtain an analytical solution for asymmetric auctions with general distributions. Nevertheless, an analytical solution can be obtained for uniform distributions.[4] Define

$$H_I = \frac{\overline{v}^2}{2\overline{v}-v_i-v_j}\left(\frac{v_I}{\overline{v}}\right)^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}}, \quad H_J = \frac{\overline{v}^2}{2\overline{v}-v_i-v_j}\left(\frac{v_J}{\overline{v}}\right)^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_j}},$$

$$\text{and} \quad H_i = \frac{\overline{v}^2}{2\overline{v}-v_i-v_j}\left(\frac{v_i}{\overline{v}}\right)^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}}.$$

The following lemma characterizes the equilibrium bidding strategy under the assumption that $F(v)$ is uniformly distributed.

Lemma 9 Consider an asymmetric all pay auction with two bidders I, J. Their valuations v_I and v_J are independently distributed. Bidder I's valuation is distributed uniformly on $[v_i, \overline{v}]$, and bidder J's valuation is distributed uniformly on $[v_j, \overline{v}]$, where $v_i \geq v_j$. The following strategies form a Bayesian Nash Equilibrium.

Player I uses bidding function $b_I(v_I; v_i, v_j) = H_I - H_i$;

Player J uses bidding function

$$b_J(v_J; v_i, v_j) = \begin{cases} H_J - H_i, & \text{if} \quad \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}} \leq v_J \leq \overline{v}; \\ 0, & \text{if} \quad v_j \leq v_J < \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}}. \end{cases} \tag{3.6}$$

In this equilibrium, the players' payoffs are given by

$$\Pi_I(v_I; v_i, v_j) = \frac{\overline{v}-v_i}{\overline{v}-v_j} H_I - \frac{v_j}{\overline{v}-v_j} v_I + H_i \;; \tag{3.7}$$

[3] Amann and Leininger [2] consider the asymmetric all pay auction with two players, Maskin and Riley [44] discuss asymmetric auctions from the aspect of revenue under more general settings.

[4] One good example can be found in Krishna [36]

$$\Pi_J(v_J; v_i, v_j) = \begin{cases} \frac{\overline{v}-v_j}{\overline{v}-v_i} H_J - \frac{v_i}{\overline{v}-v_i} v_J + H_i, & \text{if} \quad \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}} \leq v_J \leq \overline{v}; \\ 0, & \text{if} \quad v_j \leq v_J < \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}}. \end{cases} \tag{3.8}$$

Figure 1 illustrates the important features of the two bidding functions above. When $v_j < v_i$, I define player J as the weak player and I as the strong player. For the weak player, if he is sufficiently weak, he bids 0. The cutoff valuation for the weak player, who is indifferent between bidding and not bidding, lies between v_j and v_i, i.e. $v_j \leq \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}} \leq v_i$.[5] Furthermore, the weak player always bids more aggressively than the strong player.

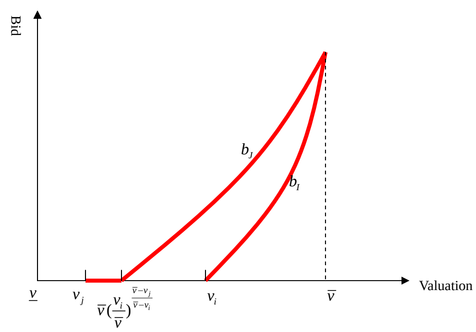

Figure 3.1: Bidding functions in an asymmetric all pay auction

When all the players followed their equilibrium strategies in the first round, the equilibrium in its non-deviated continuation game in the second round is just a straight-forward application of Lemma 9. Note that either player A or player B can be the strong player, depending on the revealed valuations of the eliminated players.

In order to characterize the equilibrium bidding strategies in the first round, I need to examine one more continuation game in the second round. In this continuation game, only one player deviated and did not following his equilibrium bidding function in the first round, but he was able to enter the second round. Without loss of generality, let player 1 be the deviated

[5] $\overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}} \leq \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_i}{\overline{v}-v_i}} = v_i$ and $\overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}} \geq \overline{v}(\frac{v_j}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_j}} = v_j$, implied by Claim 4 in the appendix

player. Suppose that player 1 bids $b^P(w)$ instead of his equilibrium bid $b^P(v_1)$. If he fails to enter the second round (i.e. $w < v_2$), then the case is irrelevant to my analysis. However, if he is able to enter the second round (i.e. $w > v_2$) and becomes player A, then he competes with player B (the winner of group B) in the second round.

In the second round, player B believes that player A's valuation is uniformly distributed on $[v_a, \overline{v}]$, while player A believes that player B's valuation is uniformly distributed on $[v_b, \overline{v}]$. However, player B's belief could be wrong, i.e. player A's true valuation v_A could lie outside of $[v_a, \overline{v}]$, as player A deviated in the first round. Define

$$H_a = \frac{\overline{v}^2}{2\overline{v} - v_a - v_b} \left(\frac{v_a}{\overline{v}}\right)^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_a}}, \quad H_b = \frac{\overline{v}^2}{2\overline{v} - v_a - v_b} \left(\frac{v_b}{\overline{v}}\right)^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_b}},$$

$$H_A = \frac{\overline{v}^2}{2\overline{v} - v_a - v_b} \left(\frac{v_A}{\overline{v}}\right)^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_a}}, \quad \text{and} \quad H_B = \frac{\overline{v}^2}{2\overline{v} - v_a - v_b} \left(\frac{v_B}{\overline{v}}\right)^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_b}},$$

The following lemma describes the strategies in this special deviated continuation game.

Lemma 10 In the special deviated continuation game in the second round described above, player B bids according to the following function:

$$b_B^F(v_B; v_a, v_b) = \begin{cases} H_B - H_b, & \text{if } v_a < v_b \text{ and } v_b \le v_B \le \overline{v}; \\ H_B - H_a, & \text{if } v_a \ge v_b \text{ and } \overline{v}(\frac{v_a}{\overline{v}})^{\frac{\overline{v} - v_b}{\overline{v} - v_a}} \le v_B \le \overline{v}; \\ 0, & \text{if } v_a \ge v_b \text{ and } v_b \le v_B \le \overline{v}(\frac{v_a}{\overline{v}})^{\frac{\overline{v} - v_b}{\overline{v} - v_a}}. \end{cases}$$

Player A bids according to the following function:

$$b_A^F(v_A; v_a, v_b) = \begin{cases} 0, & \text{if } v_a \le v_b \text{ and } \underline{v} \le v_A \le \overline{v}(\frac{v_b}{\overline{v}})^{\frac{\overline{v} - v_a}{\overline{v} - v_b}}; \\ H_A - H_b, & \text{if } v_a \le v_b \text{ and } \overline{v}(\frac{v_b}{\overline{v}})^{\frac{\overline{v} - v_a}{\overline{v} - v_b}} < v_A \le \overline{v}; \\ 0, & \text{if } v_a > v_b \text{ and } \underline{v} \le v_A \le v_a; \\ H_A - H_a, & \text{if } v_a > v_b \text{ and } v_a < v_A \le \overline{v}. \end{cases}$$

Player A's payoff is given by:

$$\Pi_A^F(v_A;v_a,v_b) \equiv \begin{cases} \Pi_1(v_A;v_a,v_b) = 0, & \text{if } v_a \le v_b \text{ and } \underline{v} \le v_A \le \overline{v}(\frac{v_b}{\overline{v}})^{\frac{\overline{v}-v_a}{\overline{v}-v_b}}; \\ \Pi_2(v_A;v_a,v_b) = \frac{\overline{v}-v_a}{\overline{v}-v_b}H_A - \frac{v_b v_A}{\overline{v}-v_b} + H_b, & \text{if } v_a \le v_b \text{ and } \overline{v}(\frac{v_b}{\overline{v}})^{\frac{\overline{v}-v_a}{\overline{v}-v_b}} < v_A \le \overline{v}; \\ \Pi_3(v_A;v_a,v_b) = v_A\frac{\overline{v}(\frac{v_a}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-v_a}} - v_b}{\overline{v}-v_b}, & \text{if } v_a > v_b \text{ and } \underline{v} \le v_A \le v_a; \\ \Pi_4(v_A;v_a,v_b) = \frac{\overline{v}-v_a}{\overline{v}-v_b}H_A - \frac{v_b v_A}{\overline{v}-v_b} + H_a, & \text{if } v_a > v_b \text{ and } v_a < v_A \le \overline{v}. \end{cases}$$

There are two important features about this continuation game. First, Player B's strategy does not depend on player A's pretended valuation, simply because he cannot observe player A's bid in the first round. Second, and because of the first feature, A's strategy also does not depend on his own pretended valuation.

I now have the necessary continuation games in the second round to analyze the first round strategies.

3.4.2 First Round Strategy

Lemma 10 gives players' strategies in the special deviated continuation game in the second round, in which only player 1 deviates in the first round and is able to enter the second round. Given this, I can formulate his optimization problem at the beginning of the first round. Suppose that all other players adopt equilibrium bidding function $b^P(\cdot)$, player 1's expected payoff when he pretends to have valuation w is given by:

$$\begin{aligned} & \max_w E\{\Pi_A^F(v_1;V_2,V_b)\mathcal{I}_{\{w>V_2\}}|v_1\} - b^P(w) \qquad (3.9) \\ = \quad & \max_w E\{\Pi_1(v_1;V_2,V_b)\mathcal{I}\{w > V_2, V_2 < V_b, v_1 < \overline{v}(\frac{V_b}{\overline{v}})^{\frac{\overline{v}-V_2}{\overline{v}-V_b}}|v_1\} \\ & + E\{\Pi_2(v_1;V_2,V_b)\mathcal{I}\{w > V_2, V_2 < V_b, v_1 > \overline{v}(\frac{V_b}{\overline{v}})^{\frac{\overline{v}-V_2}{\overline{v}-V_b}}|v_1\} \\ & + E\{\Pi_3(v_1;V_2,V_b)\mathcal{I}_{\{w>V_2,V_2>V_b,v_1<V_2\}}|v_1\} + E\{\Pi_4(v_1;V_2,V_b)\mathcal{I}_{\{w>V_2,V_2>V_b,v_1>V_2\}}|v_1\} \\ & - b^P(w) \end{aligned}$$

The first four terms represent the expected gain if he wins the prize, which depends on the realization of the eliminated bids. The last term is the cost of bidding. In equilibrium, it is optimal for the players to truthfully report their valuations. The following proposition characterizes the equilibrium bidding strategy in the first round as well as the second round.

Proposition 8 In elimination contests with partial revelation, the symmetric separating PBNE is as follows:

In the first round, all players adopt the same strictly increasing bidding function:

$$b^P(v) = \int_{\underline{v}}^{v} \frac{2\xi}{(\overline{v}-\underline{v})^3} \left\{ \frac{\overline{v}(\overline{v}-\xi)}{ln\frac{\xi}{\overline{v}}} \left[(\frac{\xi}{\overline{v}})^{\frac{\overline{v}-v}{\overline{v}-\xi}} - \frac{\xi}{\overline{v}} \right] - \frac{\xi^2-\underline{v}^2}{2} \right\} d\xi \tag{3.10}$$

In the second round, player A believes that player B's valuation is uniformly distributed on $[v_b, \overline{v}]$, while player B believes that player A's valuation is uniformly distributed on $[v_a, \overline{v}]$. They use the bidding strategy described in Lemma 9 in the second round.

As I can show in the proof, a separating equilibrium always exists. In this equilibrium, efficient of entry to the second round is achieved. However, since the second round is an asymmetric all pay auction, the allocation of the prize may not be efficient. If I examine equation (C.6), I find that the problem is no longer equivalent to a standard all pay auction, since I cannot separate the valuations from the winning probabilities. With partial information revelation, the equilibrium is partially efficient.

3.5 No Revelation

I finally analyze the case that the organizer chooses not to reveal any information after the first round of contest. The finalists know only who won and who lost in the first round. This case has been analyzed by Moldovanu and Sela [50]. I summarize their results in the following proposition.

Proposition 9 (Moldovanu and Sela (2006)) In the elimination contest with no revelation, the symmetric separating PBNE is as follows.

In the first round, players bid according to the strictly increasing function:

$$b^P(v) = \int_{\underline{v}}^{v} F(\xi)^2 \left[F(v) - F(\xi)\right] d\xi \tag{3.11}$$

In the second round, the finalists believe that their rivals' valuations are drawn from the distribution $F(\cdot)^2$ and bid according to the strictly increasing function:

$$b^F(v) = \int_{\underline{v}}^{v} \xi dF(\xi)^2 \tag{3.12}$$

If I define the value of entering the second round as a player's new valuation, then the first round contest is equivalent to a standard all pay auction. In this auction, a player's payoff is equal to his valuation of the prize multiplied by the winning probability in the second round, and minus his bid. However, under the other two information revelation rules, this is not the case. A player's valuation and winning probability are not separable.

The existence of a separating equilibrium guarantees the most efficient entry to the second round. Since the finalists are using symmetric bidding function in the second round, the prize goes to the player with the highest valuation. As a result, the outcome is efficient and the player with the highest valuation always wins the prize.

3.5.1 The Optimality of the No Revelation Rule

When the contest designer has the power to choose any information revelation rule he wishes, he would choose one that maximizes his objective. Note that the designer's decision space is very large. He can, for example, announce the average, or the sum of the bids. He can even adopt a stochastic information revelation rule, say, with probability one half announcing nothing and with probability one half announcing all bids.

Suppose that the contest designer's objective is to maximize the total efforts (bids) from the players. It is difficult to formulate all possible information revelation rules and it appears that characterizing the optimal rule is a complicated task. However, I am able to show that the no revelation rule is the optimal mechanism in a more relaxed environment. In this environment, the designer can choose both how much information to reveal and the form of competition. The allowable forms of competition include, but is not limited to, elimination contests. If I am able to prove that the elimination contest with the no revelation rule is optimal in this more general environment, then it must also be optimal in the original, more restricted environment.

Let us now describe this more relaxed environment. The designer must award one indivisible prize to one of the four players. He can decide the rule of allocating the prize and how much effort a player puts in conditional on the reported types of all players. The designer's objective is to maximize the total efforts (bids) from the players. This environment is almost identical to the optimal auction design problem in Myerson [52]. The only additional restriction is that the designer must award the prize to some player and cannot keep it for himself regardless of the information revelation rule he chooses. Given my assumption of independent private valuations, it is straightforward to show that it is optimal to award the prize to the player with the highest valuation.[6] It is easy to see that an elimination contest with the "no revelation rule" implements this exact mechanism. Thus I have the following proposition. Note that any information revelation rule that always results in the same prize allocation, i.e., the player with the highest valuation always wins the prize, is optimal.

Proposition 10 For a contest designer whose objective is to maximize the total bids (efforts) from the players, an elimination contest with the no revelation rule is optimal. Therefore, within elimination contests, the no revelation rule is optimal among all possible information revelation rules.

I have just established that an elimination contest with no revelation is optimal. However, this rule is not always feasible in real-life contests. After the first round of contests, some information may be revealed automatically due to the nature of the contests. In this case, one of the revelation rules analyzed in previous sections may apply, and the equilibrium may not be fully efficient.

3.6 Conclusion

In this essay, I consider several commonly observed information revelation rules and address how they affect the existence of separating equilibria and the allocation of the prize in elimination contests. In general, different revelation rules imply different allocations of the prize. Only the no revelation rule induces efficient entry to the second round and efficient allocation of the prize in the second round. Furthermore, it maximizes the total efforts from the players. The partial revelation rule introduces distortion of efficiency in the second round because of asymmetric information in that round, even though efficient entry to the second round is still achieved. The

[6] The effort levels are implied by the incentive compatibility constraints.

full revelation rule cannot warrant efficient entry to the second round, since there does not exist any separating equilibria with strictly increasing bidding function in the first round. Any equilibrium under the full revelation rule would necessarily involve inefficient entry to the second round competition. I show that the non-existence result is very robust with full information revelation. My analysis confirms the findings of non-existence of separating equilibrium by other researchers in other auction models with full interim information revelation.

Within the framework of my model, I can conclude that more information revelation in the interim stages reduces the efficiency of the contest. The no revelation rule guarantees that the player with the highest valuation eventually wins the prize. It is also the best rule to maximize player total efforts. Even though these conclusions are drawn from a four-player model with two rounds of competitions, the qualitative results should remain valid in a model with any arbitrary number of players and any arbitrary number of rounds of competitions. As long as the objective is to facilitate efficient entry to later rounds of competitions and to ensure that the player with the highest valuation wins the prize, the no revelation rule is optimal.

CHAPTER 4

SIMULTANEOUS SIGNALING IN ELIMINATION CONTESTS

4.1 Related Literature

Early work on eliminating contests considers the case of complete information (See Groh et al [22], Horen and Riezman [31], Hwang [35], Rosen [58], and Schwenk [59]). However, incomplete information is conventionally an interesting topic in economic theory. Moldovanu and Sela [50] consider a two-round elimination contest under incomplete information, but assume that the finalists only know that their rivals are the winners from other groups. Therefore, there is no information transmission regarding players' actions in the first round. If all the prizes are awarded to the winner in the final round, the model is equivalent to a static contest with several prizes in terms of revenue and efficiency. They point out in their paper that "an interesting avenue is to focus on the role of information in contests with multiple rounds", which is the motivation for this essay.

To my knowledge, this is the first paper to talk about the strategic impact of signalling in contests. Lai and Matros (2006) have already considered a two-round elimination contest with full revelation or no revelation of bidders' bids and characterized the corresponding equilibria. In their paper, both rounds are all-pay auctions. Their model is more general than ours, allowing for more players dividing into more groups in the first round. Furthermore, there are interim prizes for the first round winners. However, their approach is different from ours. They assume that the first-round winners are committed to act according to their pretended type in the second round in a deviation. Furthermore, a deviator's valuation also becomes that of the pretended type. That is, when a player deviates by pretending to be a different type, he becomes that pretending type in the second round.

My essay is also related to Amegashie [3]. He analyzes the signaling effect in dynamic contests with one-sided incomplete information and finds that informed players exert higher effort in the preliminary round when the opponent in the final round is weaker than they are and vice versa. In his model, both of the rounds are lotteries. Players are fooled in equilibrium because of bounded rationality. My essay investigates a two-sided incomplete information model with fully rational players.

Finally, my essay is also related to the literature on signaling in auctions, in which all players have the chance to signal and bids are made simultaneously. Goeree [21] considers an auction followed by an aftermarket, in which bidders compete for an advantage in future strategic interactions.[1] Haile [26, 27, 28] considers an auction followed by a resale auction organized by the first-round winner. Bidders have noisy private signals and their information further improves in the resale round. In all of those papers, the authors focus on the separating Perfect Bayesian Nash Equilibrium, which is also the equilibrium concept employed in my essay. However, a separating equilibrium may not exist under certain situations.[2] It is worthwhile to mention that Mailath [39] gives a sufficient condition to ensure the existence of a separating equilibrium in simultaneous signaling two-period games. Unfortunately, his result can not

[1] Also, in the introduction, he provides a good review of the literature on the signaling in auctions.

[2] Goeree [21] states "if bidders want to understate their private information, a separating equilibrium may fail to exist when the incentives to signal via a lower bid are stronger for higher valuations". Meanwhile, Haile [28] also finds that, due to the signaling effect, player's objective function is not quasiconcave, and a separating equilibrium may fail to exist.

be applied to my model directly since he assumes that the payoffs are additively separable, a condition which does not apply in my model. However, the above paper still provides us with helpful insight in the analysis.

4.2 The model

I consider a two-round elimination contest described as follows. There are $2N$ risk neutral players in the contest. All players are divided into two groups. Player 1 to N are in group A while player $N+1$ to $2N$ are in group B. Players first compete within their group in the preliminary round simultaneously. The winner of group A is denoted as player A, and the winner of group B is denoted as player B. The winners from the preliminary round (players A and B) enter the final round and compete for the prize.

Players have private information regarding their own valuation of the prize. Assume player i's valuation is V_i, with realization denoted by v_i. Although players do not know other players' valuations, they believe that they are drawn independently from a commonly known distribution $F(\cdot)$, with associated density function $f(\cdot)$ and support $\mathcal{V} = [\underline{v}, \overline{v}]$. I assume that both the valuation and the density of valuation are bounded and away from 0, i.e. $0 < \underline{v} < \overline{v} < +\infty$ and $0 < f(v) < +\infty$ for $v \in [\underline{v}, \overline{v}]$.

Player i competes with his rivals by making a bid b_i and all bids are submitted simultaneously. Regardless of success, all players pay for their bids. Therefore, a player's payoff is equal to his valuation multiplied by the winning probability less his bid. In the preliminary round the competition technology is an all pay auction: the player with the highest effort wins. In contrast, in the final round the competition technology is a lottery: a player's winning probability is equal to the ratio of his own bid to the total bid.

4.2.1 The benchmark

I assume that regardless of the outcome in the preliminary round, players' valuations become common knowledge after the preliminary round and before the final round. Though it is not clear how this mechanism is to be implemented, similar mechanisms have been used to analyze signaling in auctions with an aftermarket [21], auctions with resale [25], and collusion in auctions [48]. Here, the benchmark model is presented without worrying about the signaling effect, which will be analyzed later on.

I first describe the timing of the game:

(1) $2N$ players are equally divided into 2 groups: group A and group B.

(2) Players privately learn their valuations.

(3) A preliminary round contest is held in each group, using an all-pay auction.

(4) After the preliminary round and before the final round, players' valuations become common knowledge.

(5) The winners from the preliminary round, players A and B, compete in the final round, which is held using a lottery.

To solve the model, I employ the concept of Perfect Beyesian Nash Equilibrium (PBNE). A PBNE is a pair of strategies for each player and a posterior belief distribution, where the strategy profile is sequentially rational given the belief system and the belief is derived from the strategy profile through Bayes' rule whenever possible. Since all payers are **ex-ante** identical before learning their valuations, I look for the symmetric preliminary round bidding strategy. Within the equilibria, I focus on the separating equilibrium that, in the preliminary round, players bid according to a strictly increasing function. I first look at the final round.

Final round strategies

The winners from the preliminary round, players A and B, enter the final round. Since all players' valuations are revealed before the final round, players' beliefs about one another's valuations are not affected by preliminary round actions. As a result, the final round is a complete information game, where players simultaneously choose their bids. Throughout this essay, superscripts denote the round, P or F, and the subscripts denote the player. The winning probability is given by a simple form:

$$Prob\{player\ i\ wins\} = \frac{b_i^F}{b_i^F + b_j^F} \qquad i, j \in \{A, B\}, i \neq j$$

Player $i's$ problem in the final round is given by:

$$\max_{b_i^F}\ v_i \frac{b_i^F}{b_i^F + b_j^F} - b_i^F$$

It is useful to note that a player's payoff is a decreasing function of his rival's bid. The following lemma gives the equilibrium.

Lemma 11 In the complete information contest played by two players with valuations v_i and v_j, in which players choose bids simultaneously and the winning probability is equal to the ratio of a player's own bid to the total bids, in equilibrium, players choose their bids as follows.

$$b_i^F(v_A, v_B) = \frac{v_i^2 v_j}{(v_i + v_j)^2} \quad \forall i, j \in \{A, B\}, i \neq j$$

With the associated payoff respectively given by:

$$\Pi_i^F(v_A, v_B) = \frac{v_i^3}{(v_i + v_j)^2} \quad \forall i, j \in \{A, B\}, i \neq j$$

I refer to Nti [54] for the details of the proof.

From Lemma 11, I know that a player's bid is an increasing function of his rival's valuation when the rival's valuation is lower than his valuation, and a decreasing function of his rival's valuation when the rival's valuation is higher than his valuation.[3] This lemma is important for understanding players' strategies when they have the opportunity to signal.

Preliminary round strategies

Having solved the strategy in the final round, I move on to the preliminary round. I assume that all players adopt the same strictly increasing bidding strategy $b^P(\cdot)$. In equilibrium, a player with the lower bound valuation bids zero since he has a zero probability of winning. A player will not bid more than $b^P(\overline{v})$ since bidding $b^P(\overline{v})$ gives him the same winning probability while saving the cost. Therefore, choosing a bid to maximize one's payoff is equivalent to report one's type optimally. Since players are **ex-ante** identical, I choose player 1 as a representative. Given that all other players adopt $b^P(\cdot)$, player 1's problem at the beginning of the preliminary round is given as follows:

$$\Pi_1^P(v_1) = \max_w E\{\Pi_1^F(v_1, V_B)\mathcal{I}_{\{w>V_i, \forall\ i=2,\cdots,N\}}|v_1\} - b^P(w)$$

where $\mathcal{I}_{\{\cdot\}}$ is an indicator function and throughout the chapter all the expectations are taken on random variables, i.e the upper case letters.

The first term is the expected gain from bidding and the second term is the cost of bidding. If player 1 loses in the preliminary round, he gains nothing; if he wins the preliminary round, he gains if he also wins the final round and the payoff in the final round is given by $\Pi_1^F(v_1, v_B) = \frac{v_1^3}{(v_1+v_B)^2}$ from Lemma 11. Meanwhile, player 1 knows that player B is the win-

[3] Formally, $\frac{\partial b_i^F(v_A,v_B)}{\partial v_j} = \frac{v_i^2(v_i-v_j)}{(v_i+v_j)^3}$. When $v_j < v_i$, $\frac{\partial b_i^F(v_A,v_B)}{\partial v_j} > 0$; when $v_j > v_i$, $\frac{\partial b_i^F(v_A,v_B)}{\partial v_j} < 0$.

ner of group B. Since the preliminary round is an all-pay auction, the winner must be the one with the highest valuation in group B. For instance, player 1, if he can enter the final round, believes that player B's valuation is the first order statistic among all the players in group B, i.e. with *cdf* $F(v_B)^N$. Hence, player 1's problem turns out to be:

$$\Pi_1^P(v_1) = \max_w E\{\frac{v_1^3}{(v_1+V_B)^2}\mathcal{I}_{\{w>V_i,\forall\ i=2,\cdots,N\}}|v_1\} - b^P(w)$$

$$= \max_w E\{\mathcal{I}_{\{w>V_i,\forall\ i=2,\cdots,N\}}\}E\{\frac{v_1^3}{(v_1+V_B)^2}|v_1\} - b^P(w)$$

$$= \max_w\ F(w)^{N-1}\int_{\underline{v}}^{\overline{v}}\frac{v_1^3}{(v_1+v_B)^2}dF(v_B)^N - b^P(w)$$

I can interpret this function as follows. The term $F(w)^{N-1}$ is the winning probability in the preliminary round, the term $\int_{\underline{v}}^{\overline{v}}\frac{v_1^3}{(v_1+v_B)^2}dF(v_B)^N$ is the expected valuation of entering the final round, and the term $b^P(w)$ is his bid. Note that the payoff from losing is zero in the model setup. The following proposition gives the equilibrium bidding strategy in the preliminary round and a summary of the equilibrium of the whole game.

Proposition 11 In the elimination contest excluding the signaling effect, in which players' valuations are automatically revealed after the preliminary round and before the final round, the separating symmetric PBNE is as follows.

In the preliminary round, all players bid according to a strictly increasing bidding function:

$$b^P_{nosignaling}(v) = \int_{\underline{v}}^{v}\int_{\underline{v}}^{\overline{v}}\frac{\xi^3}{(\xi+\zeta)^2}dF(\zeta)^N dF(\xi)^{N-1}$$

In the final round, the winners from the preliminary round, players A and B, knowing each other's valuation, bid as described by Lemma 11.

Proof: see the appendix

The expected valuation of entering the final round just depends on a player's own valuation and the distribution of valuation. A high valuation player evaluates the final round higher than a low valuation player does. So if I redefine the valuation as the expected valuation of entering the final round, then the preliminary round is the same as a normal form all-pay auction. The equilibrium exists under any distribution of valuation. It is easy to check that the bidding function is indeed strictly increasing, which is consistent with my presumption.

Under this setting, preliminary round bids only affect the winning probability in the preliminary round. In contrast, as I can see below, in the game with signaling, action in the preliminary round has an extra effect. There is an inferential impact via the other players' final round strategies, since they infer the player's valuation from his action.

4.2.2 Incorporating signaling effects

I need to replace time 4 in the previous model which excludes the signaling effect:

4. After the preliminary round and before the final round begins, all bids in the preliminary round are observed by all the players and become common knowledge.

Types are not revealed automatically; instead, players' bids in the preliminary round are revealed. If I assume the equilibrium is separating and all players bid according to a strictly increasing function in the preliminary round, say $b^P(\cdot)$, then after the bids are revealed players can infer other players' valuations by inverting the bidding function. The final round is a complete information game and coincides with the final round in the model without signaling effects. However, since the finalists are informed of the bids in the preliminary round, and valuations remain unobservable to them, it is conceivable that a player may want to disguise himself by over-representing or under-representing his valuation in the preliminary round in order to gain some advantages in the final round. To solve the model, I employ the concept of PBNE and work backwards.

Final round strategies

Suppose that in the preliminary round all players adopt the same strictly increasing bidding function, $b^P(\cdot)$. I denote the image of the preliminary round strategy as $b^P(\mathcal{V}) = [b^P(\underline{v}), b^P(\overline{v})]$. It is not hard to see that the lower bound of $b^P(\mathcal{V})$ is $b^P(\underline{v}) = 0$, since the player with the lowest valuation has zero probability of entering the final round and thus would not bid more than zero. If a player bids $b \in b^P(\mathcal{V})$ in the preliminary round, then according to Bayes' rule, other players in the final round believe that his valuation is $(b^P)^{-1}(b)$. If a player's bid is outside the image of the preliminary round bidding function, i.e. $b > b^P(\overline{v})$,[4] I assume that all other players believe that he has valuation $\overline{v}$. As a result, bidding $b^P(\overline{v})$ strictly dominates bidding $b > b^P(\overline{v})$, since there is no benefit for the higher bid while the cost is higher. Under this specification of the off-path beliefs, no player has any incentive to deviate to a valuation outside the support of the valuation space $\mathcal{V}$. Therefore, I will focus on deviations of bids within the image of the preliminary round bidding function.

In the non-deviated continuation game, where all players follow the equilibrium strategy, the final round becomes a lottery with complete information and coincides with the model without the signaling effect. In order to characterize the equilibrium bidding strategies in the preliminary round, I need to examine one more deviated continuation game in the final round. In this continuation game, only one player deviates and does not following his equilibrium bidding function in the preliminary round, but is able to enter the final round.

[4] Since players can not bid a negative amount, the only possible bid, which is outside the image of the preliminary round bidding function, is $b > b^P(\overline{v})$.

Since all players are **ex-ante** symmetric, I choose player 1 as the representative player and assume that he is the one who deviates. Let $w = (b^P)^{-1}(b) \in \mathcal{V}$ be the valuation other players believe player 1 has, which may or may not be player 1's true valuation. If he loses in the preliminary round, then it does not affect the final round contest. The following analysis applies when he wins and enters the final round, and thus becomes player A.

In the final round, player B, the winner from group B, infers that player A's valuation is w. Player A learns that player B's valuation is v_B from his preliminary round bid. Furthermore, player A knows that player B believes that he has valuation w. Of course, player A knows that his own valuation is actually v_A. The following lemma describes the equilibrium in this special deviated continuation game.

Lemma 12 In the special deviated continuation game in the final round described above, player B bids:

$$b_B^F(w, v_B) = \frac{v_B^2 w}{(w + v_B)^2}$$

and player A bids:

$$b_A^F(w, v_A, v_B) = \begin{cases} 0 & if\ w \geq v_A\ and\ v_B \geq \frac{w\sqrt{v_A}}{\sqrt{w}-\sqrt{v_A}} \\ \\ \frac{v_B\sqrt{wv_A}}{w+v_B} - \frac{v_B^2 w}{(w+v_B)^2} & otherwise \end{cases}$$

and player A's associated payoff is:

$$\Pi_A^F(w, v_A, v_B) = \begin{cases} 0 & if\ w \geq v_A\ and\ v_B \geq \frac{w\sqrt{v_A}}{\sqrt{w}-\sqrt{v_A}} \\ \\ (\sqrt{v_A} - \frac{v_B\sqrt{w}}{w+v_B})^2 & otherwise \end{cases}$$

Proof: See appendix

When player A bluffs in the preliminary round and meets a very strong rival in the final round, I may have a corner solution; it is optimal for him to drop out in the competition and bid zero. Otherwise, his bid in the final round is an interior solution.

Preliminary Round Strategies

I now consider the preliminary round bidding function. Lemma 12 gives players' strategies in the special deviated continuation game when player 1 deviates and enters the final round. Obviously, player 1's expected surplus from entering the final round depends on both his true valuation and his pretended valuation. Suppose that in the preliminary round all other players adopt the equilibrium bidding function $b^P(\cdot)$ and player 1 has valuation v_1 but pretends to have valuation w. Then player 1' payoff in the whole game is given by:

$$E\{\Pi_1^F(w, v_1, V_B)\mathcal{I}_{\{w>V_i, \forall\ i=2,\cdots,N\}}|v_1\} - b^P(w)$$

$$= E\{\mathcal{I}_{\{w>V_i, \forall\ i=2,\cdots,N\}}\}E\{\Pi_1^F(w, v_1, V_B)|v_1\} - b^P(w)$$

The first term $E\{\mathcal{I}_{\{w>V_i, \forall\ i=2,\cdots,N\}}\}$ is the winning probability in the preliminary round, the second term $E\{\Pi_1^F(w, v_1, V_B)|v_1\}$ is the expected valuation of entering the final round, and the last term is the bid.

If he chooses to under-represent his valuation ($w \leq v_1$) in the preliminary round, then his problem becomes:

$$\max_w \ F(w)^{N-1} \int_{\underline{v}}^{\overline{v}} (\sqrt{v_1} - \frac{v_B\sqrt{w}}{w + v_B})^2 dF(v_B)^N - b^P(w)$$

If he chooses to over-represent his valuation in the preliminary round, ($w \geq v_1$), then his problem becomes:

$$\max_w \ F(w)^{N-1} \int_{\underline{v}}^{min\left\{\overline{v},\ \frac{w\sqrt{v_1}}{\sqrt{w}-\sqrt{v_1}}\right\}} (\sqrt{v_1} - \frac{v_B\sqrt{w}}{w + v_B})^2 dF(v_B)^N - b^P(w)$$

However, if he just over-represents his valuation locally, i.e. by an infinitly small amount, then his problem becomes:

$$\max_{w} \; F(w)^{N-1} \int_{\underline{v}}^{\overline{v}} (\sqrt{v_1} - \frac{v_B\sqrt{w}}{w+v_B})^2 dF(v_B)^N - b^P(w)$$

A weak player anticipates that if he can enter the final round he has a greater chance to meet a player stronger than him. In that case, he would like to sandbag in the preliminary round and induce his rival in the final round to underestimate him and to bid less. If this signaling effect dominates the effect of “sandbagging” decreasing the winning probability, then it is always better to sandbagin the preliminary round and truthful reporting would not be the optimal choice. As a result, a separating equilibrium may not exist. I need some restrictions on the distribution of valuation to exclude this situation.

Assumption 1 Let $G(w, v_1)$ be defined by $F(w)^{N-1} \int_{\underline{v}}^{\overline{v}} (\sqrt{v_1} - \frac{v_B\sqrt{w}}{w+v_B})^2 dF(v_B)^N$. I assume that:

(i) $G_1(v_1, v_1) > 0, \; \forall v_1 \in (\underline{v}, \overline{v}]$

(ii) $G_{12}(w, v_1) > 0, \; \forall w, v_1 \in (\underline{v}, \overline{v}]$

As in many dynamic models with asymmetric information, necessary and sufficient conditions that ensure existence are difficult to identify. Here, I give only one sufficient condition. Restriction (i) ensures that $b^P(\cdot)$ is strictly increasing, and restriction (ii) is the single-crossing condition.[5]

The restrictions (i) and (ii) given in the above assumption seem to be complicated and people may doubt the existence of such a distribution of valuation that can satisfy both restric-

[5] A more straightforward formulation of the single crossing condition is as follows. Define $U(v_1, w, b^P) = G(w, v_1) - b^P$, where v_1 is player’s true valuation and w is his perceived valuation when he bids b^P. Single crossing condition implies that: $U_{b^P}(v_1, w, b^P)/U_w(v_1, w, b^P) = \frac{-1}{G_1(w,v_1)}$ is increasing in v_1, i.e. the ratio of the marginal cost of signaling to the marginal benefit of signaling is lower for higher valuations.

tions. The following lemma is a sufficient condition to ensure that the distribution of valuation satisfies the restrictions above. It is straightforward since it just depends on the minimum valuation $\underline{v}$, the ratio of the maximum valuation to the minimum valuation $R = \frac{\overline{v}}{\underline{v}}$, and the minimum of the density function $M = \min_v f(v)$.

Lemma 13 The restrictions (i) and (ii) in Assumption 1 are satisfied if the distribution of valuation satisfies the following conditions:

- $1 < R < \sqrt[3]{4}$
- $(N-1)M\underline{v} > max\{\frac{(R-1)R^4}{2}, \frac{R^2-R}{8-4R^{\frac{3}{2}}}\}$

Proof: see appendix

These conditions are most likely to be valid in a political campaign. The value of winning a campaign is usually quite large for any party, and their valuations would not differ too much.

The following proposition gives the equilibrium bidding strategy in the preliminary round and a summary of the equilibrium of the whole game, under Assumption 1.

Proposition 12 Suppose Assumption 1 holds, then in the elimination contest with signaling effect, in which all players' bids are revealed after the preliminary round and before the final round, the separating symmetric PBNE is as follows.

In the preliminary round, players bid according to a strictly increasing bidding function:

$$\begin{aligned} b^P_{singaling}(v) &= \int_{\underline{v}}^{v}\int_{\underline{v}}^{\overline{v}} \frac{\xi^3}{(\xi+\zeta)^2} dF(\zeta)^N dF(\xi)^{N-1} \\ &+ \int_{\underline{v}}^{v}\int_{\underline{v}}^{\overline{v}} \frac{\xi\zeta(\xi-\zeta)F(\xi)^{N-1}}{(\xi+\zeta)^3} dF(\zeta)^N d\xi \end{aligned}$$

In the final round, the winners from the preliminary round, players A and B, knowing each other's valuation by inverting the bidding function, bid as described in Lemma 11.

Proof: See appendix

As I can see, the signaling effect affects the bidding strategy in the preliminary round. Since the equilibrium is a separating one, players can correctly infer their rival's valuation from his bid in the preliminary round in the equilibrium. Therefore, the final round turns into a complete information game and the strategy coincides with that in the model excluding the signaling effect.

4.3 Comparison

As I can see, the strategies in the final round are exactly the same under two different settings. Though the preliminary round strategies are different due to the signaling effect, I can see the relationship between the two. Recall that:

$$b^P_{nosignaling}{}'(v) = (N-1)F(v)^{N-2}f(v)\int_{\underline{v}}^{\overline{v}} \frac{v^3}{(v+\zeta)^2}dF(\zeta)^N \tag{4.1}$$

$$\begin{aligned} b^P_{signaling}{}'(v) &= (N-1)F(v)^{N-2}f(v)\int_{\underline{v}}^{\overline{v}} \frac{v^3}{(v+\zeta)^2}dF(\zeta)^N \\ &+F(v)^{N-1}\int_{\underline{v}}^{\overline{v}} \frac{v\zeta(v-\zeta)}{(v+\zeta)^3}dF(\zeta)^N \end{aligned} \tag{4.2}$$

I call $b^P{}'(v)$ the marginal willingness to bid. As I can see, the two equations share the same item $(N-1)F(v)^{N-2}f(v)\int_{\underline{v}}^{\overline{v}} \frac{v^3}{(v+\zeta)^2}dF(\zeta)^N$, which I call the winning probability effect; while (4.2) has an extra item: $S(v) = F(v)^{N-1}\int_{\underline{v}}^{\overline{v}} \frac{v\zeta(v-\zeta)}{(v+\zeta)^3}dF(\zeta)^N$, which I call the signaling effect. In the model excluding the signaling effect, actions in the preliminary round change the result through the winning probability, which is an increasing function of the bid. In contrast, in the model with the signaling effect, actions in the preliminary round have one additional effect: signaling effect, which is fully characterized by $S(v)$.

How the signaling effect affects a player's payoff depends on his valuation. If $S(v)$ is positive, then it means that pretending to be stronger is good for the player overall. In contrast, if $S(v)$ is negative, it means that pretending to be weaker is good for the player overall. The specification of the function $S(v)$ leads to the following crucial result.

Proposition 13 If $R < 2$, then there exists a threshold v^*, $\underline{v} < v^* < \overline{v}$, such that for a player with valuation $\underline{v} < v < v^*$, his marginal willingness to bid is lower in the presence of the signaling effect(for valuation $v = \underline{v}$, the marginal willingness to bid is the same); meanwhile, for a player with valuation $v^* < v \leq \overline{v}$, his marginal willingness to bid is higher in the presence of the signaling effect.

Proof: See appendix

See Figure 1 for an illustration.

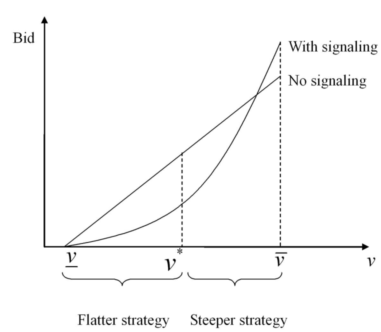

Figure 4.1: Bidding function with and without the signaling effect

Although Proposition 3 is for the case $R < 2$, I conjecture that the result holds for any distribution of valuation. Given that the distribution of valuation satisfies the restrictions in Assumption 1, the result holds.

This result is intuitive. Since a player's payoff in the final round is a decreasing function of his rival's bid, he has the incentive to disguise himself to induce the rival to bid less in the final round. Under-representing his valuation in the preliminary round makes the rival in the final round underestimate him. This has two effects. On the one hand, it makes the rival stronger than him bid less, which increases his payoff in the final round. On the other hand, it makes the rival weaker than him bid more, which decreases his payoff in the final round. Conversely, over-representing his valuation in the preliminary makes the rival in the final round overestimate him. This has two effects as well. It makes the rival stronger than him bid more, which decreases his payoff in the final round. But it also makes the rival weaker than him bid less, which decreases his payoff in the final round. In other words, if the player is strong, he anticipates that he will have a greater chance to meet a player weaker than himself in the final round, and as a result, he is willing to over-represent his valuation in the preliminary round since he wants to discourage the rival; if he is a weak player, he anticipates that he will have a greater chance to meet a player stronger than himself in the final round, and as a result, he is willing to under-represent his valuation in the preliminary round since he wants them to underestimate him.

For a strong player who has the incentive to over-report his valuation, he may end up with a lower bid compared to the no signaling case, since weak players lower their bids too much.

In the equilibrium under both settings, the winners in the preliminary are the ones with the highest valuations in their own groups, while the final rounds coincide. Furthermore, the expected payoff of a player with lower bound valuation is always zero. If I define the organizer's revenue as the total expected bids, the well-known revenue equivalence theorem would suggest that both models should generate the same revenue. In fact, revenue equivalence does not hold in my model. This is simply because the model without the signaling effect is not a feasible mechanism.

4.4 Conclusion

This chapter examines how the signaling effect works in a two-round elimination contest. Players are assumed to be **ex-ante** identical and are randomly divided into two groups. In the preliminary round, players compete within their groups and the winners enter the final round. In the benchmark model, players' valuations are automatically revealed in the final round. Thus, the expected valuation of entering the final round just depends on a player's own valuation and the distribution of valuation. In contrast, in the second model, players' valuations are not revealed in the final round while all players' bids in the preliminary round are revealed. Given that in the preliminary round players bid according to a strictly increasing function (separating equilibrium), actions in the preliminary round fully reveal players' valuations and thus affect players' actions in the final round. Since valuations are not known, players have the incentive to disguise themselves. As shown in this chapter, weak players are willing to pretend to be weaker and strong players are willing to pretend to be stronger in the presence of the signaling effect, which imposes a downward pressure on the equilibrium bidding strategy for weak players and an upward pressure for the strong players.

CHAPTER 5

CONCLUSION

The thesis studies issues in auctions and contests. Chapter 2 finds that it is of the seller's interest to provide a refund policy in order to maximize her revenue if buyers have uncertainties regarding the valuations of the object before the auction. The uncertainties are due to the idiosyncratic shocks they may learn of after purchase. Other information structures, such as common-value, information acquisition and informed principle, could also provide explanations for the reason why sellers provide refund policy and are the open questions in the future. Furthermore, the competition among sellers may induce them to provide more generous refund policy.

I only consider the elimination contests in my dissertation, but other sequential contests are popular in real life as well. The information revelation also matters in those sequential contests. In addition, another interesting future work is to look at the competition among contest designers.

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APPENDIX A

PROOFS FOR CHAPTER 2

Proof of Proposition 1
Define

$$M(b_i) = \int_{(1-K)b_i-C-w_i}^{\overline{\phi}_i} (w_i + \phi_i - b_i) dG_i(\phi_i) - \int_{\underline{\phi}_i}^{(1-K)b_i-C-w_i} (Kb_i + C) dG_i(\phi_i). \quad \text{(A.1)}$$

The function $M(b_i)$ is actually buyer i's expected surplus function if he pays b_i to the seller for the object. It can be simplified as:

$$M(b_i) = \int_{(1-K)b_i-C-w_i}^{\overline{\phi}_i} \left[w_i + \phi_i - (1-K)b_i + C\right] dG_i(\phi_i) - Kb_i - C. \quad \text{(A.2)}$$

$$\frac{dM(b_i)}{db_i} = -1 + (1-K)G_i((1-K)b_i - w_i - C) \leq 0. \quad \text{(A.3)}$$

This means that his expected surplus when he wins is decreasing in the price he pays. Now it can be shown that bidding b_i^* is a weakly dominant strategy for buyer i.

Note that it is weakly dominated to bid any amount greater than b_i^* since it gives him a nonpositive surplus even if he wins according to (A.3). Suppose now he bids an amount less than b_i^*. If he wins, he could win and get the same surplus by bidding b_i^*; if he loses, he could have won if he bids b_i^* and earns a positive surplus when the highest competing bid is less than b_i^*. Therefore, it is also weakly dominated to bid an amount less than b_i^*. In conclusion, it is a weakly dominant strategy to bid b_i^*. **Q.E.D**

Proof of Lemma 1
Equation (1) can be rewritten as:

$$\int_{(1-K)b_i^*-C-w_i}^{\overline{\phi}_i} \left[w_i + \phi_i - (1-K)b_i^* + C\right] dG_i(\phi_i) - Kb_i^* - C = 0 \quad \text{(A.4)}$$

Taking derivative with respect to C on both sides of equation (A.4) gives:

$$\begin{aligned} &\left[-(1-K)\tfrac{\partial b_i^*}{\partial C} + 1\right]\left[1 - G((1-K)b_i^* - C - w_i)\right] - 1 = 0 \\ &\Rightarrow \tfrac{\partial b_i^*}{\partial C} = \tfrac{G((1-K)b_i^*-C-w_i)}{(1-K)\left[G((1-K)b_i^*-C-w_i)-1\right]} \leq 0 \end{aligned} \quad \text{(A.5)}$$

It is strictly negative if $(1-K)b_i^* > w_i + C + \underline{\phi}_i$.

Taking derivative with respect to K on both sides of equation (A.4) gives:

$$\begin{aligned}&\left[b_i^* - (1-K)\frac{\partial b_i^*}{\partial K}\right][1 - G((1-K)b_i^* - C - w_i)] - b_i^* - K\frac{\partial b_i^*}{\partial K} = 0\\ &\Rightarrow \frac{\partial b_i^*}{\partial K} = \frac{-b_i^* G((1-K)b_i^* - C - w_i)}{1-(1-K)G((1-K)b_i^* - C - w_i)} \leq 0\end{aligned} \tag{A.6}$$

It is strictly negative if $(1-K)b_i^* > w_i + C + \underline{\phi}_i$.

If $K = C = 0$, equation (A.4) becomes:

$$\int_{b_i^* - w_i}^{\overline{\phi}_i} (w_i + \phi_i - b_i^*) dG_i(\phi_i) = 0 \tag{A.7}$$

Obviously, $b_i^* \geq w_i + \overline{\phi}_i$ are solutions. Since $\frac{\partial b_i^*}{\partial C}|_{b_i^* = w_i + \overline{\phi}_i} < 0$ and $\frac{\partial b_i^*}{\partial K}|_{b_i^* = w_i + \overline{\phi}_i} < 0$ according to (A.5) and (A.6), those are the only solutions.

If $Kw_i + C \geq -\underline{\phi}_i$, $b_i^* = w_i$ satisfies equation (A.4), since

$$\int_{(1-K)w_i - C - w_i}^{\overline{\phi}_i} [w_i + \phi_i - (1-K)w_i + C]\, dG_i(\phi_i) - Kw_i - C \tag{A.8}$$

$$= \int_{\underline{\phi}_i}^{\overline{\phi}_i} (Kw_i + \phi_i + C) dG_i(\phi_i) - Kw_i - C = 0 \tag{A.9}$$

The last step follows the fact that shocks have zero means.

If otherwise, then

$$(1-K)b_i^* \geq (1-K)w_i = (1-K)w_i + C - C = w_i + C - (Kw_i + C) > w_i + C + \underline{\phi}_i.$$

Therefore, $\frac{\partial b_i^*}{\partial C}$ and $\frac{\partial b_i^*}{\partial K}$ are strictly decreasing according to (A.5) and (A.6). **Q.E.D**

Proof for Lemma 3

$$\begin{aligned}
&\tilde{U}^i(\tilde{\phi}_i, \phi_i; \tilde{w}_i, w_i, w_{-i})\\
&= (w_i + \phi_i + C_b) y_i(\tilde{w}_i, w_{-i}, \tilde{\phi}_i) - C_b - t_i^2(\tilde{w}_i, w_{-i}, \tilde{\phi}_i)\\
&= [\tilde{w}_i + (w_i + \phi_i - \tilde{w}_i) + C_b]\, y_i(\tilde{w}_i, w_{-i}, \tilde{\phi}_i) - C_b - t_i^2(\tilde{w}_i, w_{-i}, \tilde{\phi}_i)\\
&\leq [\tilde{w}_i + (w_i + \phi_i - \tilde{w}_i) + C_b]\, y_i(\tilde{w}_i, w_{-i}, w_i + \phi_i - \tilde{w}_i) - C_b - t_i^2(\tilde{w}_i, w_{-i}, w_i + \phi_i - \tilde{w}_i)\\
&= (w_i + \phi_i + C_b) y_i(\tilde{w}_i, w_{-i}, w_i + \phi_i - \tilde{w}_i) - C_b - t_i^2(\tilde{w}_i, w_{-i}, w_i + \phi_i - \tilde{w}_i)\\
&= \tilde{U}^i(w_i + \phi_i - \tilde{w}_i, \phi_i; \tilde{w}_i, w_i, w_{-i}).
\end{aligned}$$

The inequality follows from the fact that it is optimal for a winner with initial estimate $\tilde{w}_i$ and shock $w_i + \phi_i - \tilde{w}_i$ to truthfully report his shock if has already truthfully reported his initial estimate in the first stage. From the above formula, it is optimal for the winner to report his shock as $\tilde{\phi}_i^* = w_i + \phi_i - \tilde{w}_i$ if he has reported his initial estimate as $\tilde{w}_i$.

Meanwhile,

$$\begin{aligned}
\tilde{U}^i(\tilde{\phi}_i^*, \phi_i; \tilde{w}_i, w_i, w_{-i}) &= (w_i + \phi_i + C_b) y_i(\tilde{w}_i, w_{-i}, \tilde{\phi}_i^*) - C_b - t_i^2(\tilde{w}_i, w_{-i}, \tilde{\phi}_i^*)\\
&= (\tilde{w}_i + \tilde{\phi}_i^* + C_b) y_i(\tilde{w}_i, w_{-i}, \tilde{\phi}_i^*) - C_b - t_i^2(\tilde{w}_i, w_{-i}, \tilde{\phi}_i^*)\\
&= \tilde{U}^i(\tilde{\phi}_i^*, \tilde{\phi}_i^*; \tilde{w}_i, \tilde{w}_i, w_{-i}).
\end{aligned}$$

This means that his payoff is equal to the payoff of an honest buyer with an initial estimate $\tilde{w}_i$ and a realized shock $\tilde{\phi}_i^*$. **Q.E.D**

Proof for Lemma 4

$$\begin{aligned}
&U^i(\tilde{w}_i, w_i)\\
&= \int_{\mathcal{W}_{-i}} \int_{\oplus_i} x_i(\tilde{w}_i, w_{-i}) \tilde{U}^i(\tilde{\phi}_i^*, \phi_i; \tilde{w}_i, w_i, w_{-i}) dG_i(\phi_i) dF_{-i}(w_{-i})\\
&\quad - \int_{\mathcal{W}_{-i}} t_i^1(\tilde{w}_i, w_{-i}) dF_{-i}(w_{-i})\\
&= \int_{\mathcal{W}_{-i}} \int_{\oplus_i} x_i(\tilde{w}_i, w_{-i}) \tilde{U}^i(\tilde{\phi}_i^*, \tilde{\phi}_i^*; \tilde{w}_i, \tilde{w}_i, w_{-i}) dG_i(\phi_i) dF_{-i}(w_{-i})\\
&\quad - \int_{\mathcal{W}_{-i}} t_i^1(\tilde{w}_i, w_{-i}) dF_{-i}(w_{-i})\\
&= \int_{\mathcal{W}_{-i}} \int_{\oplus_i} x_i(\tilde{w}_i, w_{-i}) \left[\tilde{U}^i(\underline{\phi}_i, \underline{\phi}_i; \tilde{w}_i, \tilde{w}_i, w_{-i}) + \int_{\underline{\phi}_i}^{\tilde{\phi}_i^*} y_i(\tilde{w}_i, w_{-i}, \xi) d\xi\right] dG_i(\phi_i) dF_{-i}(w_{-i})\\
&\quad - \int_{\mathcal{W}_{-i}} t_i^1(\tilde{w}_i, w_{-i}) dF_{-i}(w_{-i}).
\end{aligned} \tag{A.10}$$

The first step follows equation (2.13) and the second step follows equation (2.10). If $\tilde{w}_i = w_i$, from equation (A.10), I get,

$$\begin{aligned}
&U^i(\tilde{w}_i, \tilde{w}_i)\\
&= \int_{\mathcal{W}_{-i}} \int_{\oplus_i} x_i(\tilde{w}_i, w_{-i}) \left[\tilde{U}^i(\underline{\phi}_i, \underline{\phi}_i; \tilde{w}_i, \tilde{w}_i, w_{-i}) + \int_{\underline{\phi}_i}^{\phi_i} y_i(\tilde{w}_i, w_{-i}, \xi) d\xi\right] dG_i(\phi_i) dF_{-i}(w_{-i})\\
&\quad - \int_{\mathcal{W}_{-i}} t_i^1(\tilde{w}_i, w_{-i}) dF_{-i}(w_{-i}).
\end{aligned} \tag{A.11}$$

(A.10)-(A.11) gives us,

$$\begin{aligned}
U^i(\tilde{w}_i, w_i) &= U^i(\tilde{w}_i, \tilde{w}_i) + \int_{\mathcal{W}_{-i}} \int_{\oplus_i} \left[x_i(\tilde{w}_i, w_{-i}) \int_{\phi_i}^{\tilde{\phi}_i^*} y_i(\tilde{w}_i, w_{-i}, \xi) d\xi\right] dG_i(\phi_i) dF_{-i}(w_{-i})\\
&= U^i(\tilde{w}_i, \tilde{w}_i) + \int_{\oplus_i} \int_{\phi_i}^{\tilde{\phi}_i^*} \int_{\mathcal{W}_{-i}} x_i(\tilde{w}_i, w_{-i}) y_i(\tilde{w}_i, w_{-i}, \xi) dF_{-i}(w_{-i}) d\xi dG_i(\phi_i)\\
&= U^i(\tilde{w}_i, \tilde{w}_i) + \int_{\oplus_i} \int_{\phi_i}^{\tilde{\phi}_i^*} Q_i(\tilde{w}_i, \xi) d\xi dG_i(\phi_i)\\
&= U^i(\tilde{w}_i, \tilde{w}_i) + \int_{\oplus_i} \int_{\tilde{w}_i}^{w_i} Q_i(\tilde{w}_i, z + \phi_i - \tilde{w}_i) dz dG_i(\phi_i)\\
&= U^i(\tilde{w}_i, \tilde{w}_i) + \int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(\tilde{w}_i, z + \phi_i - \tilde{w}_i) dG_i(\phi_i) dz.
\end{aligned} \tag{A.12}$$

In the second last step, I change the variable $z = \xi + \tilde{w}_i - \phi_i$. Therefore, the incentive compatibility constraints in the first stage (2.15) are equivalent to,

$$U^i(w_i, w_i) \geq U^i(\tilde{w}_i, \tilde{w}_i) + \int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(\tilde{w}_i, z + \phi_i - \tilde{w}_i) dG_i(\phi_i) dz. \tag{A.13}$$

I need to show that (A.13) and (2.16) imply (2.17), (2.18) and (2.19).

Using (A.13) twice (once with the role of $\tilde{w}_i$ and w_i switched), I get

$$\int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(\tilde{w}_i, z + \phi_i - \tilde{w}_i) dG_i(\phi_i) dz \leq U^i(w_i, w_i) - U^i(\tilde{w}_i, \tilde{w}_i)$$

$$\leq \int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(w_i, z + \phi_i - w_i) dG_i(\phi_i) dz.$$

When $w_i \geq \tilde{w}_i$, I can rewrite the above formula as follows:

$$\frac{\int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(\tilde{w}_i, z + \phi_i - \tilde{w}_i) dG_i(\phi_i) dz}{w_i - \tilde{w}_i} \leq \frac{U^i(w_i, w_i) - U^i(\tilde{w}_i, \tilde{w}_i)}{w_i - \tilde{w}_i}$$

$$\leq \frac{\int_{\tilde{w}_i}^{w_i}\int_{\oplus_i} Q_i(w_i, z+\phi_i-w_i)dG_i(\phi_i)dz}{w_i-\tilde{w}_i}.$$

It is easy to see that $\int_{\oplus_i}\int_{\tilde{w}_i}^{w_i} Q_i(\tilde{w}_i, z+\phi_i-\tilde{w}_i)dG_i(\phi_i)dz$ is bounded. Therefore, by the Lebesgue convergence theorem

$$\begin{array}{rl} & \lim_{w_i\to\tilde{w}_i} \frac{\int_{\tilde{w}_i}^{w_i}\int_{\oplus_i} Q_i(\tilde{w}_i,z+\phi_i-\tilde{w}_i)dzdG_i(\phi_i)}{w_i-\tilde{w}_i} \\ = & \int_{\oplus_i}\lim_{w_i\to\tilde{w}_i}\frac{\int_{\tilde{w}_i}^{w_i} Q_i(\tilde{w}_i,z+\phi_i-\tilde{w}_i)dz}{w_i-\tilde{w}_i}dG_i(\phi_i) \\ = & \int_{\oplus_i} Q_i(\tilde{w}_i, \tilde{w}_i+\phi_i-\tilde{w}_i)dG_i(\phi_i) \\ = & \int_{\oplus_i} Q_i(\tilde{w}_i,\phi_i)dG_i(\phi_i); \end{array}$$

by analogous reasoning,

$$\lim_{\tilde{w}_i\to w_i}\frac{\int_{\tilde{w}_i}^{w_i}\int_{\oplus_i} Q_i(w_i,z+\phi_i-w_i)dzdG_i(\phi_i)}{w_i-\tilde{w}_i} = \int_{\oplus_i} Q_i(w_i,\phi_i)dG_i(\phi_i).$$

Therefore,

$$\frac{dU^i(w_i,w_i)}{dw_i} = \int_{\oplus_i} Q_i(w_i,\phi_i)dG_i(\phi_i). \tag{A.14}$$

Since this derivative is finite for all w_i , it can be recovered from its derivative, and I obtain (2.18). The same argument applies when $w_i \leq \tilde{w}_i$.

From (A.14), I can get

$$U^i(w_i,w_i) = U^i(\tilde{w}_i,\tilde{w}_i) + \int_{\tilde{w}_i}^{w_i}\int_{\oplus_i} Q_i(z,\phi_i)dG_i(\phi_i)dz.$$

Substitute it into (A.13), I get (2.17).

Of course (2.19) follows directly from (2.16). Therefore, all the conditions in Lemma 4 follow from (A.13)and (2.16).

Now I must show that (2.17), (2.18) and (2.19) also imply (A.13) and (2.16).

Since $\int_{\underline{w}_i}^{w_i}\int_{\oplus_i} Q_i(\xi,\phi_i)dG_i(\phi_i)d\xi \geq 0$ by (2.2) and (2.3), (2.16) follows from (2.18) and (2.19).

To show (A.13), (2.17) and (2.18) give us

$$\begin{array}{rl} U^i(w_i,w_i) & = U^i(\tilde{w}_i,\tilde{w}_i) + \int_{\tilde{w}_i}^{w_i}\int_{\oplus_i} Q_i(z,\phi_i)dG_i(\phi_i)dz \\ & \geq U^i(\tilde{w}_i,\tilde{w}_i) + \int_{\tilde{w}_i}^{w_i}\int_{\oplus_i} Q_i(\tilde{w}_i,z+\phi_i-\tilde{w}_i)dG_i(\phi_i)dz. \end{array}$$

This proves the lemma. **Q.E.D**

Proof for Corollary 3

The function $U^i(w_i,w_i)$ can be formulated as follows:

$$\begin{aligned} & U^i(w_i,w_i) \\ & = \int_{\mathcal{W}_{-i}}\int_{\oplus_i}(w_i+\phi_i)x_i(w)y_i(w,\phi_i)dG_i(\phi_i)dF_{-i}(w_{-i}) \\ & \quad - C_b\int_{\mathcal{W}_{-i}}\int_{\oplus_i} x_i(w)\left[1-y_i(w,\phi_i)\right]dG_i(\phi_i)dF_{-i}(w_{-i}) \\ & \quad - \int_{\mathcal{W}_{-i}}\int_{\oplus_i} x_i(w)t_i^2(w,\phi_i)dG_i(\phi_i)dF_{-i}(w_{-i}) - \int_{\mathcal{W}_{-i}} t_i^1(w)dF_{-i}(w_{-i}) \\ & = \int_{\mathcal{W}_{-i}}\int_{\oplus_i}(w_i+\phi_i+C_b)x_i(w)y_i(w,\phi_i)dG_i(\phi_i)dF_{-i}(w_{-i}) - C_b\int_{\mathcal{W}_{-i}} x_i(w)dF_{-i}(w_{-i}) \\ & \quad \underbrace{- \int_{\mathcal{W}_{-i}}\int_{\oplus_i} x_i(w)t_i^2(w,\phi_i)dG_i(\phi_i)dF_{-i}(w_{-i}) - \int_{\mathcal{W}_{-i}} t_i^1(w)dF_{-i}(w_{-i})}_{-T_i(w_i)}. \end{aligned} \tag{A.15}$$

In the first equation, the first line is the expected utility of consuming the object, the second line is the expected cost if he wins but has to return the object, and the third line is the sum of the total expected money transfers in both stages.

Rearrange the terms and use the notation in equations (2.4) and (2.5), I get:

$$\begin{aligned} t_i^1(w)dF_{-i}(w_{-i}) \quad &= \int_{\oplus_i}(w_i+\phi_i+C_b)Q_i(w_i,\phi_i)dG_i(\phi_i) - C_bX_i(w_i) - U^i(\underline{w}_i,\underline{w}_i) \\ &- \int_{\underline{w}_i}^{w_i}\int_{\oplus_i} Q_i(\xi,\phi_i)d\phi_i - \int_{\mathcal{W}_{-i}}\int_{\oplus_i} x_i(w)t_i^2(w,\phi_i)dG_i(\phi_i)dF_{-i}(w_{-i})d\xi. \end{aligned} \tag{A.16}$$

The second step follows equation (2.18). **Q.E.D**

Proof for Proposition 2

As in the standard optimal auction design literature, I can first ignore the constraints (2.9) and (2.17), and then prove that the optimal mechanism satisfy these constraints.

From equation (A.15), I can rearrange the terms and use the notation of expectation to get:

$$\begin{aligned} T_i(w_i) \quad &= E\{(w_i+\Phi_i+C_b)Q_i(w_i,\Phi_i)\} - C_bX_i(w_i) - U^i(w_i,w_i) \\ &= E\{(w_i+\Phi_i+C_b)Q_i(w_i,\Phi_i)\} - C_bX_i(w_i) - U^i(\underline{w}_i,\underline{w}_i) - \int_{\underline{w}_i}^{w_i} E\{Q_i(\xi,\Phi_i)\}d\xi \end{aligned} \tag{A.17}$$

Throughout the appendix, expectations are taken over all upper case variables.

$$\begin{aligned} ET_i(W_i) \quad &= E\{(W_i+\Phi_i+C_b)Q_i(W_i,\Phi_i)\} - C_bEX_i(W_i) \\ &\quad - U^i(\underline{w}_i,\underline{w}_i) - \int_{\underline{w}_i}^{\overline{w}_i}\int_{\underline{w}_i}^{w_i} E\{Q_i(\xi,\Phi_i)\}d\xi dF_i(w_i) \\ &= E\{(W_i+\Phi_i+C_b)x_i(W)y_i(W,\Phi_i)\} - C_bEx_i(W) \\ &\quad - U^i(\underline{w}_i,\underline{w}_i) - \int_{\underline{w}_i}^{\overline{w}_i}\frac{1-F_i(w_i)}{f_i(w_i)}E\{Q_i(w_i,\Phi_i)\}dF_i(w_i) \\ &= E\{(W_i+\Phi_i+C_b)x_i(W)y_i(W,\Phi_i)\} - C_bEx_i(W) \\ &\quad - U^i(\underline{w}_i,\underline{w}_i) - E\{\frac{1-F_i(W_i)}{f_i(W_i)}x_i(W)y_i(W,\Phi_i)\} \\ &= E\{(W_i+\Phi_i+C_b-\frac{1-F_i(W_i)}{f_i(W_i)})x_i(W)y_i(W,\Phi_i)\} - C_bEx_i(W) - U^i(\underline{w}_i,\underline{w}_i). \end{aligned}$$

The seller's revenue is given by:

$$\begin{aligned} &\sum_{i=1}^n ET_i(W_i) + RE\{[1-\sum_{i=1}^n x_i(W)]\} \\ &+ SE\{\sum_{i=1}^n x_i(W)\,[1-y_i(W,\Phi_i)]\} - C_sE\sum_{i=1}^n x_i(W) \\ = &\sum_{i=1}^n E\{\Big[(W_i+\Phi_i+C_b-S-\tfrac{1-F_i(W_i)}{f_i(W_i)})y_i(W,\Phi_i) - C_b - R - C_s + S\Big]x_i(W)\} \\ &+ R - \sum_{i=1}^n U^i(\underline{w}_i,\underline{w}_i) \\ = &\int_{\mathcal{W}}\sum_{i=1}^n\Big\{\int_{\oplus_i}\Big[w_i+\phi_i+C_b-S-\tfrac{1-F_i(w_i)}{f_i(w_i)}\Big]y_i(w,\phi_i)dG_i(\phi_i) \\ &- C_b - R - C_s + S\Big\}x_i(w)dF(w) + R - \sum_{i=1}^n U^i(\underline{w}_i,\underline{w}_i). \end{aligned}$$

The remaining constraints are (2.2), (2.3) and (2.19). First, it is obvious that it is optimal to set $U^i(\underline{w}_i,\underline{w}_i) = 0 \;\; \forall i \in N$.

Second, it is optimal to choose $y_i(w,\phi_i)$ as follows:

$$y_i(w,\phi_i) = \begin{cases} 1 & if\ \phi_i \geq -J_i(w_i) + S - C_b \\ 0 & otherwise \end{cases}, \qquad \forall i \in N, \tag{A.18}$$

where $J_i(w_i) = w_i - \frac{1-F_i(w_i)}{f_i(w_i)}$ is the virtual initial estimate. Since the hazard rate of the initial estimate is increasing, the virtual initial estimate function $J_i(w_i)$ is also increasing.

Now the objective function becomes:

$$\int_{\mathcal{W}} \sum_{i=1}^{n} \left\{ \int_{-J_i(w_i)-C_b+S}^{\overline{\phi}_i} [\phi_i + J_i(w_i) + C_b - S]\, dG_i(\phi_i) - C_b - R - C_s + S \right\} x_i(w) dF(w) + R,$$

Therefore, it is clear that the choice of $x_i(w)$ should be as follows:

$$x_i(w) = \begin{cases} 1 & if\ i = argmax_j \left\{ \mathcal{J}_i(w_i), R + C_s + C_b - S \right\} \\ 0 & otherwise \end{cases}, \tag{A.19}$$

where $\mathcal{J}_i(w_i) = \int_{-J_j(w_j)-C_b+S}^{\overline{\phi}_j} (\phi_j + J_j(w_j) + C_b - S) dG_j(\phi_j)$ is called the modified virtual initial estimate. It is helpful to note that the modified virtual initial estimate is also increasing. To see this,

$$\mathcal{J}_i'(w_i) = \int_{-J_j(w_j)-C_b+S}^{\overline{\phi}_j} J_j'(w_j) dG_j(\phi_j) = J_j'(w_j)(1 - G(-J_j(w_j - C_b + S))) \geq 0.$$

Now I have to verify whether the constraints (2.9) and (2.17) are satisfied. It is obvious that (2.9) is satisfied from (A.18). It is much more complicated to verity (2.17). First, let us consider the case $\tilde{w}_i \leq w_i$.

For $z \geq \tilde{w}_i$,

$$x_i(z, w_{-i}) = \begin{cases} 1 & if\ i = argmax_j \left\{ \mathcal{J}_i(z), R + C_s + C_b - S \right\} \\ 0 & otherwise \end{cases},$$

$$x_i(\tilde{w}_i, w_{-i}) = \begin{cases} 1 & if\ i = argmax_j \left\{ \mathcal{J}_i(\tilde{w}_i), R + C_s + C_b - S \right\} \\ 0 & otherwise \end{cases}.$$

Since $\mathcal{J}_i(\cdot)$ is increasing, I have

$$x_i(z, w_{-i}) \geq x_i(\tilde{w}_i, w_{-i}), \tag{A.20}$$

$$y_i(z, w_{-i}, \phi_i) = \begin{cases} 1 & if\ \phi_i \geq -z + \frac{1-F_i(z)}{f_i(z)} + S - C_b \\ 0 & otherwise \end{cases}, \qquad \forall i \in N, \tag{A.21}$$

$$\begin{aligned} y_i(\tilde{w}_i, w_{-i}, z + \phi_i - \tilde{w}_i) &= \begin{cases} 1 & if\ z + \phi_i - \tilde{w}_i \geq -J_i(\tilde{w}_i) + S - C_b \\ 0 & otherwise \end{cases}, \qquad \forall i \in N \\ &= \begin{cases} 1 & if\ z + \phi_i - \tilde{w}_i \geq -\tilde{w}_i + \frac{1-F_i(\tilde{w}_i)}{f_i(\tilde{w}_i)} + S - C_b \\ 0 & otherwise \end{cases}, \forall i \in N \\ &= \begin{cases} 1 & if\ \phi_i \geq -z + \frac{1-F_i(\tilde{w}_i)}{f_i(\tilde{w}_i)} + S - C_b \\ 0 & otherwise \end{cases}, \qquad \forall i \in N. \end{aligned} \tag{A.22}$$

Since the hazard rate function is increasing, I get: $\frac{1-F_i(\tilde{w}_i)}{f_i(w_i)} \geq \frac{1-F_i(z)}{f_i(z)}$. Therefore, comparing (A.21) and (A.22), I get:

$$y_i(z, w_{-i}, \phi_i) \geq y_i(\tilde{w}_i, w_{-i}, z + \phi_i - \tilde{w}_i). \tag{A.23}$$

Combining (A.20) and (A.23), I get:

$$x_i(z, w_{-i})y_i(z, w_{-i}, \phi_i) \geq x_i(\tilde{w}_i, w_{-i})y_i(\tilde{w}_i, w_{-i}, z + \phi_i - \tilde{w}_i) \tag{A.24}$$

$$\begin{aligned} \Rightarrow \quad & \int_{\oplus_i} \int_{\mathcal{W}_{-i}} x_i(z, w_{-i})y_i(z, w_{-i}, \phi_i)dF_{-i}(w_{-i})dG_i(\phi_i) \\ & \geq \int_{\oplus_i} \int_{\mathcal{W}_{-i}} x_i(\tilde{w}_i, w_{-i})y_i(\tilde{w}_i, w_{-i}, z + \phi_i - \tilde{w}_i)dF_{-i}(w_{-i})dG_i(\phi_i) \end{aligned} \tag{A.25}$$

$$\Rightarrow \quad \int_{\oplus_i} Q_i(z, \phi_i)dG_i(\phi_i) \geq \int_{\oplus_i} Q_i(\tilde{w}_i, z + \phi_i - \tilde{w}_i)dG_i(\phi_i) \tag{A.26}$$

$$\Rightarrow \quad \int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(z, \phi_i)dG_i(\phi_i)dz \geq \int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(\tilde{w}_i, z + \phi_i - \tilde{w}_i)dG_i(\phi_i)dz. \tag{A.27}$$

(A.25) follows because (A.24) is valid for any ϕ_i and w_{-i}; (A.27) follows because (A.26) is valid for any $z \geq \tilde{w}_i$. The same argument can be used for the case $w_i \leq \tilde{w}_i$.

Given the allocation rule $y_i(w, \phi_i)$, I can pin down the second round money transfer. From equation (2.11) and (A.18),
when $\phi_i \leq -J_i(w_i) + S - C_b$, $y_i(w, \phi_i) = 0$,

$$t_i^2(w, \phi_i) = -C_b - \tilde{U}^i(\underline{\phi}_i, \underline{\phi}_i; w_i, w_i, w_{-i});$$

when $\phi_i \geq -J_i(w_i) + S - C_b$,

$$\begin{aligned} t_i^2(w, \phi_i) &= (w_i + \phi_i + C_b) - \int_{-J_i(w_i)+S-C_b}^{\phi_i} d\xi - C_b - \tilde{U}^i(\underline{\phi}_i, \underline{\phi}_i; w_i, w_i, w_{-i}) \\ &= (w_i + \phi_i + C_b) - \left[\phi_i + w_i - \frac{1-F_i(w_i)}{f_i(w_i)} - S + C_b\right] - C_b - \tilde{U}^i(\underline{\phi}_i, \underline{\phi}_i; w_i, w_i, w_{-i}) \\ &= \frac{1-F_i(w_i)}{f_i(w_i)} + S - C_b - \tilde{U}^i(\underline{\phi}_i, \underline{\phi}_i; w_i, w_i, w_{-i}). \end{aligned}$$

Potentially, the second round money transfer $t_i^2(w, \phi_i)$ not only depends on the winner's initial estimate but also all the other buyers' initial estimates through $\tilde{U}^i(\underline{\phi}_i, \underline{\phi}_i; w_i, w_i, w_{-i})$. This confirms that what information the seller should announce at the beginning of the second stage is important. It is generally not without loss of generality to assume full information revelation within stages in search of the optimal selling mechanism in the original environment.

Since I can arbitrarily choose $\tilde{U}^i(\underline{\phi}_i, \underline{\phi}_i; w_i, w_i, w_{-i})$ and it would not affect the results, I can let

$$\tilde{U}^i(\underline{\phi}_i, \underline{\phi}_i; w_i, w_i, w_{-i}) = \frac{1 - F_i(w_i)}{f_i(w_i)} + S - C_b,$$

and $t_i^2(w, \phi_i)$ turns out to be:

$$t_i^2(w, \phi_i) = \begin{cases} 0 & if\ \phi_i \geq -J_i(w_i) + S - C_b \\ -\frac{1-F_i(w_i)}{f_i(w_i)} - S & otherwise \end{cases}. \tag{A.28}$$

The first round money transfer, $t_i^1(w)$, is determined by equation (2.20). The first round money transfer matters only in terms of expectation as in Myerson's paper. This completes the prove. **Q.E.D**

Proof for Lemma 4

It is important to recall that $J_i'(w_i) \geq 0$ and $\mathcal{J}_i'(w_i) \geq 0$.
Take derivative with respect to R on both side of equation (2.25):

$$\mathcal{J}_i'(w_i)\frac{\partial \hat{w}_i}{\partial R} = 1 \Rightarrow \frac{\partial \hat{w}_i}{\partial R} \geq 0.$$

Take derivative with respect to C_s on both side of equation (2.25):

$$\mathcal{J}_i'(w_i)\frac{\partial \hat{w}_i}{\partial C_s} = 1 \Rightarrow \frac{\partial \hat{w}_i}{\partial C_s} \geq 0.$$

Take derivative with respect to C_b on both side of equation (2.25):

$$\mathcal{J}_i'(w_i)\frac{\partial \hat{w}_i}{\partial C_b} + [1 - G(-J_i(w_i) + S - C_b)] = 1 \Rightarrow \frac{\partial \hat{w}_i}{\partial C_b} \geq 0.$$

Take derivative with respect to C_b on both side of equation (2.25):

$$\mathcal{J}_i'(w_i)\frac{\partial \hat{w}_i}{\partial C_b} - [1 - G(-J_i(w_i) + S - C_b)] = -1 \Rightarrow \frac{\partial \hat{w}_i}{\partial C_b} \leq 0.$$

Take derivative with respect to R on both side of equation (2.26):

$$\frac{\partial \hat{\phi}_i}{\partial R} = 0.$$

Take derivative with respect to C_s on both side of equation (2.26):

$$\frac{\partial \hat{\phi}_i}{\partial C_s} = 0.$$

Take derivative with respect to C_b on both side of equation (2.26):

$$\frac{\partial \hat{\phi}_i}{\partial C_b} = -1 < 0.$$

Take derivative with respect to S on both side of equation (2.26):

$$\frac{\partial \hat{\phi}_i}{\partial S} = 1 > 0.$$

Q.E.D

Proof for Proposition 3
I am looking for a symmetric separating PBNE in which all buyers adopt the same strictly increasing bidding function in the auction stage. I assume this bidding function is $b_2(\cdot)$ and use backward induction to solve the model. Given $b_2(\cdot)$, I define the fee as $C_2(p,q) = p - \frac{1-F(b_2^{-1}(q))}{f(b_2^{-1}(q))} - S$, where p is the price the winner pays in the auction and q is the winner's bid in the auction. Therefore, $b_2^{-1}(q)$ is the reported initial estimate of the winner, denoted as w; $b_2^{-1}(p)$ is the highest reported initial estimate of the buyers other than the winner, denoted as

Y_1. In the return stage, the winner, say buyer i, keeps the object if and only if consuming the object is better than returning it:

$$\begin{aligned} & w_i + \phi_i \geq p - C(p,q) - C_b \\ & \Rightarrow w_i + \phi_i \geq p - \left[p - \tfrac{1-F(w)}{f(w)} - S\right] - C_b \\ & \Rightarrow w_i + \phi_i \geq p - \left[p - \tfrac{1-F(w)}{f(w)} - S\right] - C_b \\ & \Rightarrow \phi_i \geq -w_i + \tfrac{1-F(w)}{f(w)} + S - C_b. \end{aligned} \tag{A.29}$$

Giving all the other buyers truthfully report their initial estimates, I can formulate a representative's problem, say buyer i, in the auction stage. He chooses his report w to maximize his utility:

$$\begin{aligned} u(w, w_i) &= \int_{\underline{w}}^{w} \{ \int_{-w_i + \frac{1-F(w)}{f(w)} + S - C_b}^{\overline{\phi}} (w_i + \phi_i) dG(\phi_i) \\ &\quad + \int_{\underline{\phi}}^{-w_i + \frac{1-F(w)}{f(w)} + S - C_b} [b_2(Y_1) - C(b_2(Y_1), b_2(w)) - C_b] dG(\phi_i) \\ &\quad - b_2(Y_1)\} dF(Y_1)^{n-1} \\ &= \int_{\underline{w}}^{w} \{ \int_{-w_i + \frac{1-F(w)}{f(w)} + S - C_b}^{\overline{\phi}} (w_i + \phi_i) dG(\phi_i) \\ &\quad + \int_{\underline{\phi}}^{-w_i + \frac{1-F(w)}{f(w)} + S - C_b} [\tfrac{1-F(w)}{f(w)} + S - C_b] dG(\phi_i) - b_2(Y_1)\} dF(Y_1)^{n-1} \\ &= \int_{\underline{w}}^{w} \{ \int_{-w_i + \frac{1-F(w)}{f(w)} + S - C_b}^{\overline{\phi}} [w_i + \phi_i - \tfrac{1-F(w)}{f(w)} - S + C_b] dG(\phi_i) \\ &\quad + \left[\tfrac{1-F(w)}{f(w)} + S - C_b\right] - b_2(Y_1)\} dF(Y_1)^{n-1}. \end{aligned} \tag{A.30}$$

He wins the auction only if his bid is higher than the bids of all the other buyers. If his realized shock is high, he consumes the object; if his realized shock is low, he returns the object in the sense that he gets back what he paid in the auction minus the fee and the cost of sending the object back; he pays for his bid if he wins.

In order to find the equilibrium, I use the following lemma developed in McAfee and Vincent [47].

Lemma 14 Suppose $v : [a,b]^2 \to \mathcal{R}$ is twice continuously differentiable, and $v_{12}(r,x) \geq 0 \; \forall \; x, r$. Then

$$v(r,x) \leq v(x,x) \forall \; r, x \Leftrightarrow v_1(x,x) = 0 \;\; \forall \; r, x.$$

Proof: Sufficiency is straight forward. $v(r,x) \leq v(x,x)$ means $r = x$ is a global maximum of $v(r,x)$ and therefore the first order condition applies.
Necessary condition. Since $v_{12}(r,x) \geq 0$, then for $r < x$, I have $v_1(r,x) \geq v_1(r,r) = 0$; and for $r > x$, I have $v_1(r,x) \leq v_1(r,r) = 0$. Therefore, $v(r,x)$ reaches the maximum at $r = x$.

In my model, I have:

$$\frac{\partial u(w,w_i)}{\partial w_i} = F(w)^{n-1} \left[1 - G(-w_i + \tfrac{1-F(w)}{f(w)} + S - C_b)\right], \tag{A.31}$$

$$\begin{aligned} \frac{\partial^2 u(w,w_i)}{\partial w_i \partial w} &= \frac{dF(w)^{n-1}}{dw} \left[1 - G(-w_i + \tfrac{1-F(w)}{f(w)} + S - C_b)\right] \\ &\quad - F(w)^{n-1} g(-w_i + \tfrac{1-F(w)}{f(w)} + S - C_b) \underbrace{\frac{d\left[\frac{1-F(w)}{f(w)}\right]}{dw}}_{<0} > 0. \end{aligned} \tag{A.32}$$

The last step follows the assumption that the hazard rate function of the initial estimate is increasing. Therefore, from Lemma 14, the maximization problem is equivalent to:

$$\frac{\partial u(w,w_i)}{\partial w}\Big|_{w=w_i} = 0$$

$$\Rightarrow (n-1)F(w_i)^{n-2}f(w_i)b_2(w_i)$$

$$\begin{aligned}&= (n-1)F(w_i)^{n-2}f(w_i)\{\int_{-w_i+\frac{1-F(w_i)}{f(w_i)}+S-C_b}^{\overline{\phi}}[w_i+\phi_i-\tfrac{1-F(w_i)}{f(w_i)}-S+C_b]dG(\phi_i)\\&\qquad+[\tfrac{1-F(w_i)}{f(w_i)}+S-C_b]\}\\&\quad+F(w_i)^{n-1}G(-w_i+\tfrac{1-F(w_i)}{f(w_i)}+S-C_b)\frac{d[\frac{1-F(w_i)}{f(w_i)}]}{dw_i}\end{aligned} \tag{A.33}$$

$$\begin{aligned}\Rightarrow b_2(w_i) &= \{\int_{-w_i+\frac{1-F(w_i)}{f(w_i)}+S-C_b}^{\overline{\phi}}[w_i+\phi_i-\tfrac{1-F(w_i)}{f(w_i)}-S+C_b]dG(\phi_i)\\&\qquad+[\tfrac{1-F(w_i)}{f(w_i)}+S-C_b]\}\\&\quad+\tfrac{F(w_i)}{(n-1)f(w_i)}G(-w_i+\tfrac{1-F(w_i)}{f(w_i)}+S-C_b)\frac{d[\frac{1-F(w_i)}{f(w_i)}]}{dw_i}\end{aligned} \tag{A.34}$$

It confirms that $b_2(w_i) = B_2(w_i)$. In addition, it is easy to see from equation (A.33) that the bidding function is indeed strictly increasing. If the reserve price is equal to $b_2(\mathcal{J}^{-1}(R+C_s-S+C_b))$, a buyer with initial estimate lower than $\mathcal{J}^{-1}(R+C_s-S+C_b)$ finds it is optimal to stay out of the competition.

Since the money transfers are implied by the incentive compatibility constraints, there is no need to verify them. Now, I only need to show that this PBNE leads to the optimal allocation rules.

In equilibrium, from (A.29), the winner keeps the object if and only if:

$$\phi_i \geq -w_i + \frac{1-F(w_i)}{f(w_i)} + S - C_b,$$

therefore,

$$y_i(w,\phi_i) = \begin{cases} 1 & if\ \phi_i \geq -J_i(w_i) + S - C_b \\ 0 & otherwise \end{cases}, \qquad \forall i \in N. \tag{A.35}$$

Since all buyers are using the same strictly increasing bidding function in the auction stage, the object goes to the buyer with the highest initial estimate given it is higher than the reserve price $\mathcal{J}^{-1}(R+C_s-S+C_b)$, i.e.,

$$x_i(w) = \begin{cases} 1 & if\ i = argmax_j \left\{\mathcal{J}_i(w_i), R+C_s+C_b-S\right\} \\ 0 & otherwise \end{cases}.$$

This completes the proof. **Q.E.D**

Proof for Proposition 4

I am looking for a symmetric separating PBNE in which all buyers adopt the same strictly increasing bidding function in the auction stage. I assume this bidding function is $b_1(\cdot)$ and use backward induction to solve the model. Given $b_1(\cdot)$, I define the fee as $C(p) = p - \frac{1-F(b_1^{-1}(p))}{f(b_1^{-1}(p))} - S$. Therefore, $b_1^{-1}(p)$ is the reported initial estimate of the winner, denoted as w. In the return stage, the winner, say buyer i, keeps the object if and only if consuming the object is better than returning it:

$$\begin{aligned} & w_i + \phi_i \geq p - C(p) - C_b \\ & \Rightarrow w_i + \phi_i \geq b_1(w) - \left[b_1(w) - \tfrac{1-F(w)}{f(w)} - S \right] - C_b \\ & \Rightarrow w_i + \phi_i \geq b_1(w) - \left[b_1(w) - \tfrac{1-F(w)}{f(w)} - S \right] - C_b \\ & \Rightarrow \phi_i \geq -w_i + \tfrac{1-F(w)}{f(w)} + S - C_b. \end{aligned} \tag{A.36}$$

Giving all the other buyers truthfully report their initial estimates, I can formulate a representative's problem, say buyer i, in the auction stage. He chooses his report w to maximize his utility:

$$\begin{aligned} u(w, w_i) \;&= F(w)^{n-1} \{ \quad \textstyle\int_{-w_i + \frac{1-F(w)}{f(w)} + S - C_b}^{\overline{\phi}} (w_i + \phi_i) dG(\phi_i) \\ &\qquad + \textstyle\int_{\underline{\phi}}^{-w_i + \frac{1-F(w)}{f(w)} + S - C_b} [b_1(w) - C(b_1(w)) - C_b] dG(\phi_i) - b_1(w) \} \\ &= F(w)^{n-1} \{ \quad \textstyle\int_{-w_i + \frac{1-F(w)}{f(w)} + S - C_b}^{\overline{\phi}} (w_i + \phi_i) dG(\phi_i) \\ &\qquad + \textstyle\int_{\underline{\phi}}^{-w_i + \frac{1-F(w)}{f(w)} + S - C_b} [\tfrac{1-F(w)}{f(w)} + S - C_b] dG(\phi_i) - b_1(w) \} \\ &= F(w)^{n-1} \{ \quad \textstyle\int_{-w_i + \frac{1-F(w)}{f(w)} + S - C_b}^{\overline{\phi}} [w_i + \phi_i - \tfrac{1-F(w)}{f(w)} - S + C_b] dG(\phi_i) \\ &\qquad + \left[\tfrac{1-F(w)}{f(w)} + S - C_b \right] - b_1(w) \}. \end{aligned} \tag{A.37}$$

He wins the auction only if his bid is higher than the bids of all the other buyers. If his realized shock is high, he consumes the object; if his realized shock is low, he returns the object in the sense that he gets back what he paid in the auction minus the fee and the cost of sending the object back; he pays for his bid if he wins.

In my model, $\frac{\partial u(w,w_i)}{\partial w_i}$ and $\frac{\partial^2 u(w,w_i)}{\partial w_i \partial w}$ are the same as those in the second-price auction (A.31) and (A.32). Therefore, from Lemma 14, the maximization problem is equivalent to:

$$\begin{aligned} & \left.\tfrac{\partial u(w,w_i)}{\partial w}\right|_{w=w_i} = 0 \\ & \Rightarrow [F(w_i)^{n-1} b_1(w_i)]' \;= [F(w_i)^{n-1}]' \{ \textstyle\int_{-w_i + \frac{1-F(w_i)}{f(w_i)} + S - C_b}^{\overline{\phi}} [w_i + \phi_i - \tfrac{1-F(w_i)}{f(w_i)} - S + C_b] dG(\phi_i) \\ & \qquad\qquad + [\tfrac{1-F(w_i)}{f(w_i)} + S - C_b] \} \\ & \qquad\qquad + F(w_i)^{n-1} G(-w_i + \tfrac{1-F(w_i)}{f(w_i)} + S - C_b) \tfrac{d[\frac{1-F(w_i)}{f(w_i)}]}{dw_i} \end{aligned} \tag{A.38}$$

$$\begin{aligned} \Rightarrow b_1(w_i) \;&= \tfrac{1}{F(w_i)^{n-1}} \textstyle\int_{\mathcal{J}^{-1}(R+C_s-S+C_b)}^{w_i} \left\{ \mathcal{J}(\xi) + \left[\tfrac{1-F(\xi)}{f(\xi)} + S - C_b \right] \right\} dF(\xi)^{n-1} \\ &+ \tfrac{1}{F(w_i)^{n-1}} \textstyle\int_{\mathcal{J}^{-1}(R+C_s+S+C_b)}^{w_i} F(\xi)^{n-1} G(-\xi + \tfrac{1-F(\xi)}{f(\xi)} + S - C_b) d \left[\tfrac{1-F(\xi)}{f(\xi)} \right] \\ &+ \tfrac{F(\mathcal{J}^{-1}(R+C_s-S+C_b))^{n-1}}{F(w_i)^{n-1}} \mathcal{J}^{-1}(R + C_s - S + C_b). \end{aligned} \tag{A.39}$$

The last step follows the boundary condition, $b_1(\mathcal{J}^{-1}(R + C_s - S + C_b)) = \mathcal{J}^{-1}(R + C_s - S + C_b)$. It confirms that $b_1(w_i) = B(w_i)$. In addition, it is easy to see from equation (A.38) that the bidding function is indeed strictly increasing.

The rest of the proof is the same as that in Proposition 3. This completes the proof. **Q.E.D**

Proof for Lemma 5
I have

$$U^i(\tilde{w}_i, w_i) = \int_{\oplus_i} (w_i + \phi_i + C_b) Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i) - T_i(\tilde{w}_i) - C_b X_i(\tilde{w}_i), \quad \text{(A.40)}$$

and

$$U^i(\tilde{w}_i, \tilde{w}_i) = \int_{\oplus_i} (\tilde{w}_i + \phi_i + C_b) Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i) - T_i(\tilde{w}_i) - C_b X_i(\tilde{w}_i). \quad \text{(A.41)}$$

(C.5)-(C.6) gives us,

$$U^i(\tilde{w}_i, w_i) = U^i(\tilde{w}_i, \tilde{w}_i) + (w_i - \tilde{w}_i) \int_{\oplus_i} Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i).$$

Therefore, the incentive compatibility constraint (2.30) is equivalent to

$$U^i(w_i, w_i) \geq U^i(\tilde{w}_i, \tilde{w}_i) + (w_i - \tilde{w}_i) \int_{\oplus_i} Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i) \;\; \forall i \in N, \forall w_i, \tilde{w}_i \in \mathcal{W}_i. \quad \text{(A.42)}$$

I need to show that (A.42) and (2.31) implies (2.34), (2.35) and (2.36).

Using (A.42) twice (once with the role of $\tilde{w}_i$ and w_i switched), I get

$$(w_i - \tilde{w}_i) \int_{\oplus_i} Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i) \leq U^i(w_i, w_i) - U^i(\tilde{w}_i, \tilde{w}_i) \leq (w_i - \tilde{w}_i) \int_{\oplus_i} Q_i(w_i, \phi_i) dG_i(\phi_i).$$

Then (2.34) follows, when $\tilde{w}_i \leq w_i$.

These inequalities can be rewritten for any $\delta > 0$

$$\delta \int_{\oplus_i} Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i) \leq U^i(\tilde{w}_i + \delta, \tilde{w}_i + \delta) - U^i(\tilde{w}_i, \tilde{w}_i) \leq \delta \int_{\oplus_i} Q_i(\tilde{w}_i + \delta, \phi_i) dG_i(\phi_i).$$

Since $\int_{\oplus_i} Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i)$ is increasing in $\tilde{w}_i$, it is Riemann integrable, so:

$$\int_{\underline{w}_i}^{w_i} \int_{\oplus_i} Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i) d\tilde{w}_i = U^i(w_i, w_i) - U^i(\underline{w}_i, \underline{w}_i),$$

which gives us (2.35).

Of course (2.36) follows directly from (2.31), so all the conditions in Lemma 5 follow from feasibility.

Now I must show that (2.34), (2.35) and (2.36) also imply (A.42) and (2.31).

Since $\int_{\underline{w}_i}^{w_i} \int_{\oplus_i} Q_i(\xi, \phi_i) dG_i(\phi_i) d\xi \geq 0$ by (2.2) and (2.3), (2.31) follows from (2.35) and (2.36)

To show (A.42), suppose $\tilde{w}_i \leq w_i$; then (2.34) and (2.35) give us:

$$\begin{aligned} U^i(w_i, w_i) &= U^i(\tilde{w}_i, \tilde{w}_i) + \textstyle\int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(\xi, \phi_i) dG_i(\phi_i) d\xi \\ &\geq U^i(\tilde{w}_i, \tilde{w}_i) + \textstyle\int_{\tilde{w}_i}^{w_i} \int_{\oplus_i} Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i) d\xi \\ &= U^i(\tilde{w}_i, \tilde{w}_i) + (w_i - \tilde{w}_i) \textstyle\int_{\oplus_i} Q_i(\tilde{w}_i, \phi_i) dG_i(\phi_i); \end{aligned}$$

similarly, if $\tilde{w}_i > w_i$, then,

$$\begin{aligned} U^i(w_i,w_i) &= U^i(\tilde{w}_i,\tilde{w}_i) - \int_{w_i}^{\tilde{w}_i}\int_{\oplus_i} Q_i(\xi,\phi_i)dG_i(\phi_i)d\xi \\ &\geq U^i(\tilde{w}_i,\tilde{w}_i) - \int_{w_i}^{\tilde{w}_i}\int_{\oplus_i} Q_i(\tilde{w}_i,\phi_i)dG_i(\phi_i)d\xi \\ &= U^i(\tilde{w}_i,\tilde{w}_i) + (w_i-\tilde{w}_i)\int_{\oplus_i} Q_i(\tilde{w}_i,\phi_i)dG_i(\phi_i). \end{aligned}$$

Thus (A.42) follows from (2.34) and (2.35). So the conditions in Lemma 5 also imply feasibility. This proves the lemma. **Q.E.D**

Proof for Proposition 5

Using equation (2.29), evaluating $\tilde{w}_i = w_i$, and (2.35), I can pin down the expected payment function:

$$\hat{T}_i(w_i) \quad = E\{(w_i+\Phi_i+C_b)Q_i(w_i,\Phi_i)\} - C_bX_i(w_i) - U^i(\underline{w}_i,\underline{w}_i) - \int_{\underline{w}_i}^{w_i} E\{Q_i(\xi,\Phi_i)\}d\xi.$$

It coincides the expected payment function (A.17) in the two-stage direct mechanisms. Therefore, following exactly the same algebra as in the proof of Proposition 2, I end up with the same revenue function.

$$\begin{aligned} \int_{\mathcal{W}}\sum_{i=1}^n\Big\{\int_{\oplus_i}\Big[w_i+\phi_i+C_b-S-\tfrac{1-F_i(w_i)}{f_i(w_i)}\Big]y_i(w,\phi_i)dG_i(\phi_i) \\ -C_b-R-C_s+S\Big\}x_i(w)dF(w)+R-\sum_{i=1}^n U^i(\underline{w}_i,\underline{w}_i). \end{aligned}$$

Now, the remaining constraints are (2.32), (2.33), (2.34) and (2.36). If I ignore the constraint (2.34), the problem becomes exactly the same as that in the two-stage direct mechanism after dropping constraints (2.9) and (2.17). Therefore, the solution is

$$x_i(w) = \begin{cases} 1 & if\ i = argmax_j \Big\{\mathcal{J}_i(w_i), R+C_s+C_b-S\Big\} \\ 0 & otherwise \end{cases}, \tag{A.43}$$

$$y_i(w,\phi_i) = \begin{cases} 1 & if\ \phi_i \geq -J_i(w_i)+S-C_b \\ 0 & otherwise \end{cases}. \tag{A.44}$$

Now, I only need to verify that this mechanism satisfies (2.34).

$$x_i(w)y_i(w,\phi_i) = \begin{cases} 1 & if\ i = argmax_j \Big\{\mathcal{J}_i(w_i), R+C_s+C_b-S\Big\}\ and\ \phi_i \geq -J_i(w_i)+S-C_b \\ 0 & otherwise \end{cases}$$

therefore,

$$\int_{\oplus_i} Q_i(w_i,\phi_i)dG_i(\phi_i)$$

$$= \begin{cases} [1-G(-J_i(w_i)+S-C_b)]\prod_{j\neq i}F_j(\mathcal{J}_j^{-1}(\mathcal{J}_i(w_i))) & if\ w_i \geq \mathcal{J}_i^{-1}(R+C_s+C_b-S) \\ 0 & otherwise \end{cases}$$

which is nondecreasing in w_i since $\mathcal{J}_j(w_j)$ is an nondecreasing function in w_j for all j. Therefore, (2.34) is satisfied. The allocation rules, (A.43) and (A.44), are optimal in the relaxed environment and are the same as the optimal allocation rules in the optimal two-stage direct mechanism. Since buyers' expected money transfers and the seller's revenue are determined by the allocation rules, they are the same as those in the optimal two-stage direct mechanism. In conclusion, the two-stage direct mechanism with the optimal refund policy is optimal in the relaxed environment. **Q.E.D**

APPENDIX B

PROOFS FOR CHAPTER 3

The following theorem will be used several times in the proof. A more general version of the lemma can be found in Guesnerie and Laffont [24]. However, special cases were used in several papers, such as McAfee et al [47] and Myerson [52].

Theorem 1 Consider a general all-pay auction

$$\max_v U(v, v_1) = \max_v G(v, v_1) - b(v) \tag{B.1}$$

with boundary condition $b(\underline{v}) = 0$. The expected gain function $G(v, v_1)$: $[\underline{v}, \overline{v}] \times [\underline{v}, \overline{v}] \to \mathcal{R}$ is twice continuously differentiable and depends on both the true valuation v_1 and the reported valuation v. Assume that $G_1(v_1, v_1) > 0$, $\forall v_1 \in (\underline{v}, \overline{v}]$, and $G_{12}(v, v_1) > 0$, $\forall v, v_1 \in (\underline{v}, \overline{v}]$. Then $b(v_1) = \int_{\underline{v}}^{v_1} G_1(\xi, \xi) d\xi$ is the equilibrium bidding function. This bidding function is strictly increasing and fully valuation revealing.

Proof: Suppose there exists a strictly increasing function $b(v)$ such that truthfully reporting $(v = v_1)$ is optimal, then incentive compatibility implies that

$$G_1(v_1, v_1) - b'(v_1) = 0 \Rightarrow b'(v_1) = G_1(v_1, v_1) \Rightarrow b(v_1) = \int_{\underline{v}}^{v_1} G_1(\xi, \xi) d\xi \tag{B.2}$$

Note that $b'(v_1) = G_1(v_1, v_1) > 0, \forall v_1 \in (\underline{v}, \overline{v}]$. So it is true that $b(v_1)$ is strictly increasing. Given $b(v_1) = \int_{\underline{v}}^{v_1} G_1(\xi, \xi) d\xi$, I have

$$\frac{dU(v, v_1)}{dv} = G_1(v, v_1) - b'(v) = G_1(v, v_1) - G_1(v, v) \tag{B.3}$$

Since $G_{12}(v, v_1) > 0 \forall v, v_1 \in (\underline{v}, \overline{v}]$, then $G_1(v, v_1)$ is a strictly increasing function of v_1. Hence, $\frac{dU(v,v_1)}{dv} = G_1(v, v_1) - G_1(v, v) > 0$ if $v_1 > v$; and $\frac{dU(v,v_1)}{dv} = G_1(v, v_1) - G_1(v, v) < 0$ if $v_1 < v$. Thus, $v = v_1$ is optimal. **Q.E.D**

Proof of Lemma 12: I divide the second round continuation game into the following four situations: (1) $w > v_A, w > v_B$, (2) $w > v_A, w \le v_B$, (3) $w \le v_A, w > v_B$, and (4) $w \le v_A, w \le v_B$.
Case 1: $w > v_A, w > v_B$
Player A overbids in the first round and meets a rival weaker than his pretended valuation in

the second round. Then from Lemma 6, player B randomizes his bid in the interval $[0, v_B]$ according to $c.d.f.\ G_B(b_B) = \frac{w-v_B+b_B}{w}$. Knowing player B's reaction function as well as his own true valuation, player A's optimization problem becomes

$$\max_{b_A} \Pi_A^F = \max_{b_A} v_A G_B(b_A) - b_A = \max_{b_A} v_A \frac{w - v_B + b_A}{w} - b_A$$

The first order condition give us $\frac{\partial \Pi_A^F}{\partial b_A} = \frac{v_A}{w} - 1 < 0$. Note that player A is the strong player and always wins in a tie. Therefore, it is optimal for him to bid 0 and get an expected Payoff of $\frac{v_A(w-v_B)}{w}$.

Case 2: $w > v_A, w \leq v_B$

Player A overbids in the first round and meets a rival stronger than his pretended valuation in the second round. Then from Lemma 6, player B randomizes his bid in the interval $[0, w]$ according to $c.d.f.\ G_B(b_B) = \frac{b_B}{w}$. Then player A's optimization problem becomes:

$$\max_{b_A} \Pi_A^F = \max_{b_A} v_A G_B(b_A) - b_A = \max_{b_A} v_A \frac{b_A}{w} - b_A$$

FOC: $\frac{\partial \Pi_A^F}{\partial b_A} = \frac{v_A}{w} - 1 < 0$, then it is optimal for him to bid 0 and get an expected payoff 0.

Case 3: $w \leq v_A, w > v_B$

Player A underbids in the first round and meets a rival weaker than his pretended valuation in the second round. Then from Lemma 6, player B randomizes his bid in the interval $[0, v_B]$ according to $c.d.f.\ G_B(b_B) = \frac{w-v_B+b_B}{w}$. Then player A's optimization problem becomes:

$$\max_{b_A} \Pi_A^F = \max_{b_A} v_A G_B(b_A) - b_A = \max_{b_A} v_A \frac{w - v_B + b_A}{w} - b_A$$

FOC: $\frac{\partial \Pi_A^F}{\partial b_A} = \frac{v_A}{w} - 1 > 0$, then it is optimal for him to bid v_B and get an expected payoff of $v_A - v_B$.

Case 4: $w \leq v_A, w \leq v_B$

Player A underbids in the first round and meets a rival stronger than his pretended valuation in the second round. Then from Lemma 6, player B randomizes his bid on the interval $[0, w]$ according to $c.d.f.\ G_B(b_B) = \frac{b_B}{w}$. Then player A's optimization problem becomes:

$$\max_{b_A} \Pi_A^F = \max_{b_A} v_A G_B(b_A) - b_A = \max_{b_A} v_A \frac{b_A}{w} - b_A$$

FOC: $\frac{\partial \Pi_A^F}{\partial b_A} = \frac{v_A}{w} - 1 > 0$, then it is optimal for him to bid w and get an expected payoff of $v_A - w$. I have now covered all possible cases. **Q.E.D**

Proof of Proposition 6: If player 1 chooses to overbid in the first round ($w \geq v_1$), then player 1's payoff is given by:

$$\begin{aligned} \Pi_1^P(w, v_1) &= F(w) \int_{\underline{v}}^{w} \frac{v_1(w - v_B)}{w} dF(v_B)^2 - b^P(w) \\ &= v_1 F(w)^3 - \frac{F(w)v_1}{w} \int_{\underline{v}}^{w} v_B dF(v_B)^2 - b^P(w) \end{aligned}$$

In order to ensure truthful revelation of valuation in equilibrium, the following conditions must be satisfied.

$$\left.\frac{\partial \Pi_1^P(w,v_1)}{\partial w}\right|_{w=v_1}$$

$$= 3v_1F(v_1)^2f(v_1) - \left[\frac{f(v_1)v_1^2 - F(v_1)v_1}{v_1^2}\int_{\underline{v}}^{v_1} v_B dF(v_B)^2 + \frac{F(v_1)v_1}{v_1}2v_1F(v_1)f(v_1)\right] - b^{P'}(v_1)$$
$$= v_1F(v_1)^2f(v_1) - \left[f(v_1) - \frac{F(v_1)}{v_1}\right]\int_{\underline{v}}^{v_1} v_B dF(v_B)^2 - b^{P'}(v_1) \leq 0$$

$$\Rightarrow b^{P'}(v_1) \geq \left[v_1F(v_1)^2f(v_1) - f(v_1)\int_{\underline{v}}^{v_1} v_B dF(v_B)^2\right] + \underbrace{\frac{F(v_1)}{v_1}\int_{\underline{v}}^{v_1} v_B dF(v_B)^2}_{\geq 0} \quad \text{(B.4)}$$

If player 1 chooses to underbid in the first round ($w \leq v_1$), then his payoff is given by:

$$\begin{aligned}\Pi_1^P(w,v_1) &= F(w)\int_{\underline{v}}^{w}(v_1 - v_B)dF(v_B)^2 + F(w)\int_{w}^{\overline{v}}(v_1 - w)dF(v_B)^2 - b^P(w)\\ &= v_1F(w)^3 - F(w)\int_{\underline{v}}^{w} v_B dF(v_B)^2 + F(w)(v_1 - w)[1 - F(w)^2] - b^P(w)\end{aligned}$$

In order to ensure truthful revelation of valuation in equilibrium, I also need:

$$\left.\frac{\partial \Pi_1^P(w,v_1)}{\partial w}\right|_{w=v_1}$$
$$= 3v_1F(v_1)^2f(v_1) - f(v_1)\int_{\underline{v}}^{v_1} v_B dF(v_B)^2$$
$$-2v_1F(v_1)^2f(v_1) + \left[f(v_1) - 3F(v_1)^2f(v_1)\right](v_1 - v_1) - \left[F(v_1) - F(v_1)^3\right] - b^{P'}(v_1)$$
$$= v_1F(v_1)^2f(v_1) - f(v_1)\int_{\underline{v}}^{v_1} v_B dF(v_B)^2 - \left[F(v_1) - F(v_1)^3\right] - b^{P'}(v_1) \geq 0$$

$$\Rightarrow b^{P'}(v_1) \leq \left[v_1F(v_1)^2f(v_1) - f(v_1)\int_{\underline{v}}^{v_1} v_B dF(v_B)^2\right] - \underbrace{\left[F(v_1) - F(v_1)^3\right]}_{\geq 0} \quad \text{(B.5)}$$

However, it is easy to see that equation (B.4) and (B.5) cannot be satisfied at the same time for all values of $v_1 \in [\underline{v}, \overline{v}]$. Hence, there does not exist a strictly increasing bidding function in the first round in equilibrium. Therefore, no separating equilibrium with strictly increasing bidding function exists. **Q.E.D**

Proof of Proposition 7: If player 1 chooses to overbid in the first round ($w \geq v_1$), then player 1's payoff is given by:

$$\Pi_1^P(w,v_1)$$
$$= F(w)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{v}}^{w+\phi_1-\phi_B}\frac{(v_1+\phi_1)(w+\phi_1-v_B-\phi_B)}{w+\phi_1}dF(v_B)^2dH(\phi_1)dH(\phi_B) - b^P(w)$$

In order to ensure truthful revelation of valuation in equilibrium, the following conditions must be satisfied.

$$\begin{aligned}&\left.\frac{\partial \Pi_1^P(w,v_1)}{\partial w}\right|_{w=v_1}\\ =\ & f(v_1)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{v}}^{v_1+\phi_1-\phi_B}\frac{(v_1+\phi_1)(v_1+\phi_1-v_B-\phi_B)}{v_1+\phi_1}dF(v_B)^2dH(\phi_1)dH(\phi_B)\\ & +F(v_1)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{v}}^{v_1+\phi_1-\phi_B}\frac{v_B+\phi_B}{v_1+\phi_1}dF(v_B)^2dH(\phi_1)dH(\phi_B) - b^{P'}(v_1) \leq 0\end{aligned}$$

$$\Rightarrow \quad b^{P'}(v_1) \geq f(v_1)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{v}}^{v_1+\phi_1-\phi_B}(v_1+\phi_1-v_B-\phi_B)dF(v_B)^2dH(\phi_1)dH(\phi_B)$$
$$\underbrace{+F(v_1)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{v}}^{v_1+\phi_1-\phi_B}\frac{v_B+\phi_B}{v_1+\phi_1}dF(v_B)^2dH(\phi_1)dH(\phi_B)}_{\geq 0}$$

If player 1 chooses to underbid in the first round ($w \leq v_1$), then his payoff is given by:

$$\begin{aligned}
&\Pi_1^P(w,v_1)\\
= \; &F(w)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{v}}^{w+\phi_1-\phi_B}(v_1+\phi_1-v_B-\phi_B)dF(v_B)^2dH(\phi_1)dH(\phi_B)\\
&+F(w)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}\int_{w+\phi_1-\phi_B}^{\overline{v}}(v_1-w)dF(v_B)^2dH(\phi_1)dH(\phi_B)-b^P(w)
\end{aligned}$$

In order to ensure truthful revelation of valuation in equilibrium, I also need:

$$\begin{aligned}
&\frac{\partial \Pi_1^P(w,v_1)}{\partial w}\Big|_{w=v_1}\\
&= f(v_1)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{v}}^{v_1+\phi_1-\phi_B}(v_1+\phi_1-v_b-\phi_B)dF(v_B)^2dH(\phi_1)dH(\phi_B)\\
&-F(v_1)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}[1-F(v_1+\phi_1-\phi_B)^2]dH(\phi_1)dH(\phi_B)-b^{P'}(v_1)
\end{aligned}$$

$$\Rightarrow \quad b^{P'}(v_1) \leq f(v_1)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{v}}^{v_1+\phi_1-\phi_B}(v_1+\phi_1-v_B-\phi_B)dF(v_B)^2dH(\phi_1)dH(\phi_B)$$
$$\underbrace{-F(v_1)\int_{\underline{\phi}}^{\overline{\phi}}\int_{\underline{\phi}}^{\overline{\phi}}[1-F(v_1+\phi_1-\phi_B)^2]dH(\phi_1)dH(\phi_B)}_{\leq 0}$$

However, it is easy to see that those two equations cannot be satisfied at the same time for all values of $v_1 \in [\underline{v}, \overline{v}]$. Hence, there does not exist a strictly increasing equilibrium bidding function in the first round. Therefore, no separating equilibrium with a symmetric strictly increasing bidding function exists. **Q.E.D**

Proof of Lemma 9: Suppose that in the equilibrium players I and J follow increasing bidding strategies $b_I(\cdot)$ and $b_J(\cdot)$, respectively.

Claim 1 $b_I(v_i) = b_J(v_j) = 0$

Proof: Suppose that $b_J(v_j) > 0$. Then for player I with any valuation, bidding 0 dominates bidding $b \in (0, b_J(v_j))$. This is because bidding 0 gives him payoff 0 while bidding $b \in (0, b_J(v_j))$ gives him a negative payoff. Thus, player I will not bid $b \in (0, b_J(v_j))$. However, in this case, player J with valuation v_j will not bid $b_J(v_j)$, since bidding a little bit lower is better. A similar argument can be applied to show that $b_I(v_i) = 0$. **Q.E.D**

Claim 2 $b_J(\overline{v}) = b_I(\overline{v}) = \overline{b}$

Proof: Suppose that $b_J(\overline{v}) > b_I(\overline{v})$. Then player J with valuation $\overline{v}$ is always better off to bid a little bit less, since his winning probability does not change while he saves on the cost. A similar argument applies to $b_J(\overline{v}) < b_I(\overline{v})$. **Q.E.D**

Suppose I now assume additionally that the bidding functions take the following forms. $b_I(v_I)$ is strictly increasing on its support. $b_J(v_J) = 0$ for $v_i \leq v_J \leq v^*$ and is strictly increasing on $v^* \leq v_J \leq \overline{v}$. Denote $\phi_J = b_J^{-1}$ for $v^* \leq v_J \leq \overline{v}$ and $\phi_I = b_I^{-1}$, respectively.

Player I's optimization problem is given by:

$$\max_b v_I Prob\{b > b_J(v_J)\} - b \max_b v_I Prob\{v_J < \phi_J(b)\} - b = \max_b v_I \frac{\phi_J(b) - v_j}{\overline{v} - v_j} - b$$

The first order condition gives us $\frac{1}{\overline{v}-v_j}\phi'_J(b)v_I = 1$. In equilibrium $v_I = \phi_I(b)$. Thus this FOC becomes

$$\phi'_J(b)\phi_I(b) = \overline{v} - v_j. \tag{B.6}$$

Similarly, when $v^* \leq v_J \leq \overline{v}$, from player J's optimization problem I obtain

$$\phi'_I(b)\phi_J(b) = \overline{v} - v_i \tag{B.7}$$

Adding equations (B.6) and (B.7) together, I obtain

$$\phi'_J(b)\phi_I(b) + \phi'_I(b)\phi_J(b) = 2\overline{v} - v_i - v_j \Rightarrow \frac{d(\phi_I(b)\phi_J(b))}{db} = 2\overline{v} - v_i - v_j$$

$$\Rightarrow \phi_I(b)\phi_J(b) = (2\overline{v} - v_i - v_j)b + A \tag{B.8}$$

Since $b_I(\overline{v}) = b_J(\overline{v}) = \overline{b}$, I determine the constant term in equation (B.8) as $A = \overline{v}^2 - (2\overline{v} - v_i - v_j)\overline{b}$. Thus, I have:

$$\phi_I(b)\phi_J(b) = (2\overline{v} - v_i - v_j)(b - \overline{b}) + \overline{v}^2 \tag{B.9}$$

From equations (B.7) and (B.9), I have

$$\frac{\phi'_I(b)}{\phi_I(b)} = \frac{\overline{v} - v_i}{(2\overline{v} - v_i - v_j)(b - \overline{b}) + \overline{v}^2} \tag{B.10}$$

$$\Rightarrow \phi_I(b) = B\left[(2\overline{v} - v_i - v_j)(b - \overline{b}) + \overline{v}^2)\right]^{\frac{\overline{v}-v_i}{2\overline{v}-v_i-v_j}} \tag{B.11}$$

From $b_I(\overline{v}) = \overline{b}$, I have $B = \overline{v}^{\frac{v_i - v_j}{2\overline{v}-v_i-v_j}}$. Thus, equation (B.11) becomes

$$\phi_I(b) = \overline{v}\left[\frac{2\overline{v} - v_i - v_j}{\overline{v}^2}(b - \overline{b}) + 1\right]^{\frac{\overline{v}-v_i}{2\overline{v}-v_i-v_j}} \tag{B.12}$$

Meanwhile, I have $b_I(v_i) = 0$; i.e., $\phi_I(0) = v_i$. Thus, equation (B.12) becomes

$$v_i = \overline{v}[\frac{2\overline{v} - v_i - v_j}{\overline{v}^2}(0 - \overline{b}) + 1]^{\frac{\overline{v}-v_i}{2\overline{v}-v_i-v_j}} \tag{B.13}$$

$$\Rightarrow \overline{b} = \frac{\overline{v}^2 \left[1 - (\frac{v_i}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}}\right]}{2\overline{v} - v_i - v_j} \tag{B.14}$$

Substituting equation (B.14) into equation (B.12), I get

$$\phi_I(b) = \overline{v}\left[\frac{2\overline{v} - v_i - v_j}{\overline{v}^2} b + (\frac{v_i}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}}\right]^{\frac{\overline{v}-v_i}{2\overline{v}-v_i-v_j}} \tag{B.15}$$

$$\Rightarrow b_I(v_I) = \frac{\overline{v}^2}{2\overline{v} - v_i - v_j}\left[(\frac{v_I}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}} - (\frac{v_i}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}}\right] \tag{B.16}$$

From equations (B.9), (B.14) and (B.15), I get

$$\phi_J(b) = \overline{v}\left[\frac{2\overline{v} - v_i - v_j}{\overline{v}^2} b + (\frac{v_i}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}}\right]^{\frac{\overline{v}-v_j}{2\overline{v}-v_i-v_j}} \tag{B.17}$$

$$\Rightarrow b_J(v_J) = \frac{\overline{v}^2}{2\overline{v} - v_i - v_j}\left[(\frac{v_J}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_j}} - (\frac{v_i}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}}\right] \tag{B.18}$$

Finding v^* by setting $b_J(v^*) = 0$, I have $v^* = \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}}$.
To summarize:

$$b_I(v_I) = \frac{\overline{v}^2}{2\overline{v} - v_i - v_j}\left[(\frac{v_I}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}} - (\frac{v_i}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}}\right] \tag{B.19}$$

$$b_J(v_J) = \begin{cases} \frac{\overline{v}^2}{2\overline{v}-v_i-v_j}\left[\frac{v_J}{\overline{v}}\right]^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_j}} - (\frac{v_i}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}}) & \text{if } \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}} \leq v_J \leq \overline{v} \\ 0 & \text{if } v_j \leq v_J < \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}} \end{cases} \tag{B.20}$$

I now calculate the payoffs:

$$\begin{aligned} \Pi_I(v_I) &= v_I \tfrac{\phi_J(b_I(v_I))-v_j}{\overline{v}-v_j} - b_I(v_I) \\ &= \tfrac{\overline{v}^2(\overline{v}-v_i)}{(\overline{v}-v_j)(2\overline{v}-v_i-v_j)}(\tfrac{v_I}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}} - \tfrac{v_j}{\overline{v}-v_j}v_I + \tfrac{\overline{v}^2}{2\overline{v}-v_i-v_j}(\tfrac{v_i}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_j}} \end{aligned} \tag{B.21}$$

For player J, if $v_j \leq v_J < \overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}}$, then $\Pi_J(v_J) = 0$.

If $\overline{v}(\frac{v_i}{\overline{v}})^{\frac{\overline{v}-v_j}{\overline{v}-v_i}} \leq v_J \leq \overline{v}$, then

$$\begin{aligned} \Pi_J(v_J) &= v_J \tfrac{\phi_I(b_J(v_J))-v_i}{\overline{v}-v_i} - b_J(v_J) \\ &= \tfrac{\overline{v}^2(\overline{v}-v_j)}{(\overline{v}-v_i)(2\overline{v}-v_i-v_j)}(\tfrac{v_J}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_j}} - \tfrac{v_i}{\overline{v}-v_i}v_J + \tfrac{\overline{v}^2}{2\overline{v}-v_i-v_j}(\tfrac{v_i}{\overline{v}})^{\frac{2\overline{v}-v_i-v_j}{\overline{v}-v_i}} \end{aligned} \tag{B.22}$$

Q.E.D

Proof of Lemma 10: For player B, he has followed his equilibrium strategy in the first round. Meanwhile, he believes that all the other players have also followed their equilibrium strategies in the first round. Thus, he believes that player A's valuation is uniformly distributed on $[v_a, \overline{v}]$ with $c.d.f.$ $\frac{v_A - v_a}{\overline{v} - v_a}$, and that player A considers him as a player with a valuation uniformly distributed on $[v_a, \overline{v}]$ with $c.d.f.$ $\frac{v_B - v_b}{\overline{v} - v_b}$. Note that two situations may happen in the second round: one is $v_a < v_b$ and the other is $v_a \geq v_b$. Hence, player B's strategy can be derived directly from Lemma 9.

$$b_B^F(v_B; v_a, v_b) = \tag{B.23}$$

$$\begin{cases} \frac{\overline{v}^2}{2\overline{v} - v_a - v_b}\left[(\frac{v_B}{\overline{v}})^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_b}} - (\frac{v_b}{\overline{v}})^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_b}} \right] & \text{if } v_a < v_b \text{ and } v_b \leq v_B \leq \overline{v} \\ \frac{\overline{v}^2}{2\overline{v} - v_a - v_b}\left[(\frac{v_B}{\overline{v}})^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_b}} - (\frac{v_a}{\overline{v}})^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_a}} \right] & \text{if } v_a \geq v_b \text{ and } \overline{v}(\frac{v_a}{\overline{v}})^{\frac{\overline{v} - v_b}{\overline{v} - v_a}} \leq v_B \leq \overline{v} \\ 0 & \text{if } v_a \geq v_b \text{ and } v_b \leq v_B \leq \overline{v}(\frac{v_a}{\overline{v}})^{\frac{\overline{v} - v_b}{\overline{v} - v_a}} \end{cases} \tag{B.24}$$

An important feature is that player B's strategy does not depend on player A's pretended valuation in the first round simply because player B does not know it.

For player A, he acts optimally in the second round, taking player B's bidding function as given and knowing his pretended valuation w and true valuation v_A. Given that player 1 is in the second round and becomes player A, his optimization problem in the second round is given by

$$\max_b v_A Prob\{b > b_B^F(V_B; v_a, v_b)\} - b. \tag{B.25}$$

If player A's true valuation $v_A \in [v_a, \overline{v}]$, his optimal reaction and payoff are implied by Lemma 9, since player B's belief about player A's valuation is correct.

It could also be the case that $v_A \notin [v_a, \overline{v}]$, i.e., $v_A < v_a$. I consider two situations. First, if player A is the weak player $v_a < v_b$, then it is optimal for him to bid 0, since bidding a positive amount gives him a negative payoff. This can be seen as follows:

$$\begin{aligned} v_A Prob\{b > b_B^F(V_B; v_a, v_b)\} - b &< v_a Prob\{b > b_B^F(V_B; v_a, v_b)\} - b \\ &\leq v_a Prob\{b > b_B^F(V_B; v_a, v_b)\} - b\Big|_{b=0} = 0 \end{aligned}$$

The second step is from the fact that it is optimal for the player with valuation v_a to bid zero and get payoff 0.

Second, if player A is the strong player $v_a > v_b$,

$$(B.25) = \max_b v_A \frac{\overline{v}[\frac{2\overline{v} - v_a - v_b}{\overline{v}^2} b + (\frac{v_a}{\overline{v}})^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_a}}]^{\frac{\overline{v} - v_b}{2\overline{v} - v_a - v_b}} - v_b}{\overline{v} - v_b} - b$$

Taking the derivative with respect to b gives us

$$\begin{aligned} & \frac{v_A \overline{v}}{\overline{v} - v_b} \frac{\overline{v} - v_b}{2\overline{v} - v_a - v_b} \frac{2\overline{v} - v_a - v_b}{\overline{v}^2} [\frac{2\overline{v} - v_a - v_b}{\overline{v}^2} b + (\frac{v_a}{\overline{v}})^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_a}}]^{\frac{v_a - \overline{v}}{2\overline{v} - v_a - v_b}} - 1 \\ = \ & \frac{v_A}{\overline{v}} [\frac{2\overline{v} - v_a - v_b}{\overline{v}^2} b + (\frac{v_a}{\overline{v}})^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_a}}]^{\frac{v_a - \overline{v}}{2\overline{v} - v_a - v_b}} - 1 < \frac{v_A}{\overline{v}} [(\frac{v_a}{\overline{v}})^{\frac{2\overline{v} - v_a - v_b}{\overline{v} - v_a}}]^{\frac{v_a - \overline{v}}{2\overline{v} - v_a - v_b}} - 1 \\ < \ & \frac{v_A}{v_a} - 1 < 0 \end{aligned}$$

Note that player A is the strong player and he wins whenever there is a tie. Therefore, it is optimal for him to bid 0 and get a payoff of $v_A \frac{(\frac{v_a}{\overline{v}})^{\frac{\overline{v} - v_b}{\overline{v} - v_a}} - v_b}{\overline{v} - v_b}$. Lemma 9 summarizes all the situations together. **Q.E.D**

Proof of Proposition 8: The following claims will be used several times in the proof.

Claim 3 $K(x) = xlnx + 1 - x \geq 0$ for $x \in (0, 1]$.

Claim 4 $(\frac{v_1}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-v_1}}$ is increasing in v_1, where $0 \leq v_1, v_b \leq \overline{v}$.

I will use Theorem 1 to determine the equilibrium. Player 1's problem is given by:

$$\max_w \Pi_1^P(w, v_1) = G(w, v_1) - b^P(w) \tag{B.26}$$

$$\begin{aligned} = \max\nolimits_w \quad & E\{\Pi_1(v_1; V_2, V_b)\mathcal{I}\{w > V_2, V_2 < V_b, v_1 < \overline{v}(\tfrac{V_b}{\overline{v}})^{\frac{\overline{v}-V_2}{\overline{v}-V_b}}|v_1\} \\ & + E\{\Pi_2(v_1; V_2, V_b)\mathcal{I}\{w > V_2, V_2 < V_b, v_1 > \overline{v}(\tfrac{V_b}{\overline{v}})^{\frac{\overline{v}-V_2}{\overline{v}-V_b}}|v_1\} \\ & + E\{\Pi_3(v_1; V_2, V_b)\mathcal{I}_{\{w>V_2, V_2>V_b, v_1<V_2\}}|v_1\} \\ & + E\{\Pi_4(v_1; V_2, V_b)\mathcal{I}_{\{w>V_2, V_2>V_b, v_1>V_2\}}|v_1\} - b^P(w) \end{aligned} \tag{B.27}$$

$$= \max_w E\{\Pi_1(v_1; V_2, V_b)\mathcal{I}\{w > V_2, V_2 < V_b, V_2 > \overline{v} - (\overline{v} - V_b)\frac{ln\frac{v_1}{\overline{v}}}{ln\frac{V_b}{\overline{v}}}|v_1\} \tag{B.28}$$

$$+E\{\Pi_2(v_1; V_2, V_b)\mathcal{I}\{w > V_2, V_2 < V_b, V_2 < \overline{v} - (\overline{v} - V_b)\frac{ln\frac{v_1}{\overline{v}}}{ln\frac{V_b}{\overline{v}}}|v_1\} \tag{B.29}$$

$$+E\{\Pi_3(v_1; V_2, V_b)\mathcal{I}_{\{w>V_2, V_2>V_b, v_1<V_2\}}|v_1\} \tag{B.30}$$

$$+E\{\Pi_4(v_1; V_2, V_b)\mathcal{I}_{\{w>V_2, V_2>V_b, v_1>V_2\}}|v_1\} \tag{B.31}$$

$$-b^P(w) \tag{B.32}$$

where $\Pi_1(v_1; V_2, V_b)$,$\Pi_2(v_1; V_2, V_b)$, $\Pi_3(v_1; V_2, V_b)$ and $\Pi_4(v_1; V_2, V_b)$ are defined in Lemma 10. V_2 has $c.d.f.$ $F(v_2) = \frac{v_2 - \underline{v}}{\overline{v} - \underline{v}}$. I know that V_b is the loser in group B, whose $c.d.f.$ is given by $2F(v_b) - F(v_b)^2$, where $F(v_b) = \frac{v_b - \underline{v}}{\overline{v} - \underline{v}}$.

Note that term (B.28) is always 0, since $\Pi_1(v_1; V_2, V_b) = 0$.

In order to pin down term (B.29), I need to investigate the properties of $v_1 = \overline{v}(\frac{V_b}{\overline{v}})^{\frac{\overline{v}-V_2}{\overline{v}-V_b}}$. I can rearrange it and write V_2 as a function of V_b: $V_2 = h(V_b; v_1) = \overline{v} - (\overline{v} - V_b)\frac{ln\frac{v_1}{\overline{v}}}{ln\frac{V_b}{\overline{v}}}$. Figure B.1 characterizes the important features of function $h(V_b; v_1)$.

First, it is a decreasing function:

$$\frac{\partial h(V_b; v_1)}{\partial V_b} = \frac{ln\frac{v_1}{\overline{v}}}{ln\frac{V_b}{\overline{v}}} + (\overline{v} - V_b)\frac{ln\frac{v_1}{\overline{v}}}{(ln\frac{V_b}{\overline{v}})^2}\frac{1}{V_b} = \frac{\overline{v}ln\frac{v_1}{\overline{v}}(\frac{V_b}{\overline{v}}ln\frac{V_b}{\overline{v}} + 1 - \frac{V_b}{\overline{v}})}{V_b(ln\frac{V_b}{\overline{v}})^2} < 0$$

The last step comes from the fact that $\frac{V_b}{\overline{v}}ln\frac{V_b}{\overline{v}} + 1 - \frac{V_b}{\overline{v}} > 0$, which is implied by Claim 3, and $ln\frac{v_1}{\overline{v}} < 0$.

Second, it crosses the line $V_2 = V_b$ exactly once at $V_2 = V_b = V_1$

Note that I can also express V_b as a function of V_2, $V_b = h^{-1}(V_2; v_1)$, an inverse function of $h(V_b; v_1)$.

Let us first examine the case in which player 1 underbids in the first round, i.e., $w < v_1$.

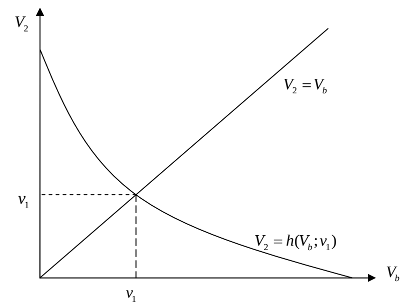

Figure B.1: Properties of the function $V_2 = h(V_b; v_1)$

From Figure B.2, I obtain

Term (B.29)= $\int_{\underline{v}}^{w} \left[\int_{v_2}^{h^{-1}(v_2;v_1)} \Pi_2(v_1; v_2, v_b) d(2F(v_b) - F(v_b)^2) \right] dF(v_2)$.

It is easy to see that Term (B.30)= 0, since $\mathcal{I}_{\{w>V_2, V_2>V_b, v_1<V_2\}} = 0$.

Term (B.31)= $\int_{\underline{v}}^{w} \left[\int_{\underline{v}}^{v_2} \Pi_4(v_1; v_2, v_b) d(2F(v_b) - F(v_b)^2) \right] dF(v_2)$.

Second, I examine the case in which player 2 over bids in the first round, i.e., $w \geq v_1$. From Figure B.3, I have

Term (B.29)=$\int_{\underline{v}}^{v_1} \left[\int_{\underline{v}}^{h^{-1}(v_2;v_1)} \Pi_2(v_1; v_2, v_b) d(2F(v_b) - F(v_b)^2) \right] dF(v_2)$.

It is easy to see that:

Term (B.30)= $\int_{v_1}^{w} \left[\int_{\underline{v}}^{v_2} \Pi_3(v_1; v_2, v_b) d(2F(v_b) - F(v_b)^2) \right] dF(v_2)$

Term (B.31)= $\int_{\underline{v}}^{v_1} \left[\int_{\underline{v}}^{v_2} \Pi_4(v_1; v_2, v_b) d(2F(v_b) - F(v_b)^2) \right] dF(v_2)$

Therefore, $G(w, v_1)$ becomes

for $w < v_1$,

$$\begin{aligned} G(w, v_1) &= \int_{\underline{v}}^{w} \left[\int_{v_2}^{h^{-1}(v_2;v_1)} \Pi_2(v_1; v_2, v_b) d[2F(v_b) - F(v_b)^2] \right] dF(v_2) \\ &\quad + \int_{\underline{v}}^{w} \left[\int_{\underline{v}}^{v_2} \Pi_4(v_1; v_2, v_b) d[2F(v_b) - F(v_b)^2] \right] dF(v_2), \end{aligned}$$

And for $w \geq v_1$,

$$\begin{aligned} G(w, v_1) &= \int_{\underline{v}}^{v_1} \left[\int_{\underline{v}}^{h^{-1}(v_2;v_1)} \Pi_2(v_1; v_2, v_b) d[2F(v_b) - F(v_b)^2] \right] dF(v_2) \\ &\quad + \int_{v_1}^{w} \left[\int_{\underline{v}}^{v_2} \Pi_3(v_1; v_2, v_b) d(2F(v_b) - F(v_b)^2) \right] dF(v_2) \\ &\quad + \int_{\underline{v}}^{v_1} \left[\int_{\underline{v}}^{v_2} \Pi_4(v_1; v_2, v_b) d(2F(v_b) - F(v_b)^2) \right] dF(v_2). \end{aligned} \tag{B.33}$$

From Theorem 1, if $G(w, v_1)$ satisfies the conditions: $G_1(v_1, v_1) > 0, G_{12}(w, v_1) > 0, \forall w, v_1 \in (\underline{v}, \overline{v}]$, then $b^P(v_1) = \int_{\underline{v}}^{v_1} G_1(\xi, \xi) d\xi$ consists an equilibrium.

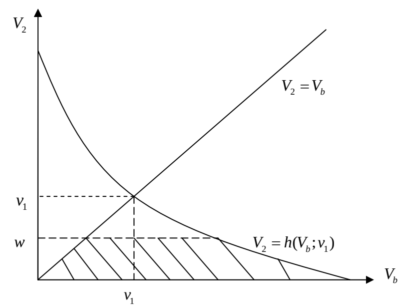

Figure B.2: Integration area for Term B.29 when $w < v_1$

For $w < v_1$, I have

$$\begin{aligned} G_1(v_1,v_1) &= f(v_1)\int_{v_1}^{h^{-1}(v_1;v_1)} \Pi_2(v_1;v_1,v_b)d[2F(v_b)-F(v_b)^2] \\ &+f(v_1)\int_{\underline{v}}^{v_1} \Pi_4(v_1;v_1,v_b)d[2F(v_b)-F(v_b)^2] \\ &= f(v_1)\int_{v_1}^{v_1} \Pi_2(v_1;v_1,v_b)d[2F(v_b)-F(v_b)^2] \\ &+f(v_1)\int_{\underline{v}}^{v_1} v_1\frac{\overline{v}(\frac{v_1}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-v_1}}-v_b}{\overline{v}-v_b}d[2F(v_b)-F(v_b)^2] \\ &= f(v_1)\int_{\underline{v}}^{v_1} v_1\frac{\overline{v}(\frac{v_1}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-v_1}}-v_b}{\overline{v}-v_b}2f(v_b)[1-F(v_b)]dv_b \end{aligned} \tag{B.34}$$

Note that $\Pi_4(v_1,v_1,v_b) = \Pi_3(v_1,v_1,v_b)$

$$\begin{aligned} G_{12}(w,v_1) &= f(w)\int_{w}^{h^{-1}(w;v_1)} \frac{\partial\Pi_2(v_1;w,v_b)}{\partial v_1}d[2F(v_b)-F(v_b)^2] \\ &+f(w)\frac{\partial h^{-1}(w;v_1)}{\partial v_1}\Pi_2(v_1;w,h^{-1}(w;v_1))2[1-F(h^{-1}(w;v_1))]f(h^{-1}(w;v_1)) \\ &+f(w)\int_{\underline{v}}^{w} \frac{\partial\Pi_4(v_1;w,v_b)}{\partial v_1}d[2F(v_b)-F(v_b)^2] \end{aligned} \tag{B.35}$$

For $w \geq v_1$, I have

$$\begin{aligned} G_1(v_1,v_1) &= f(v_1)\int_{\underline{v}}^{v_1} \Pi_3(v_1;v_1,v_b)d[2F(v_b)-F(v_b)^2] \\ &= f(v_1)\int_{\underline{v}}^{v_1} v_1\frac{\overline{v}(\frac{v_1}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-v_1}}-v_b}{\overline{v}-v_b}2f(v_b)[1-F(v_b)]dv_b \end{aligned} \tag{B.36}$$

$$G_{12}(w,v_1) = f(w)\int_{\underline{v}}^{w} \frac{\partial\Pi_3(v_1;w,v_b)}{\partial v_1}d[2F(v_b)-F(v_b)^2] \tag{B.37}$$

From this, I end up with the same $G_1(v_1,v_1)$ function and

$$G_1(v_1,v_1) \quad > f(v_1)\int_{\underline{v}}^{v_1} v_1\frac{\overline{v}(\frac{v_b}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-v_b}}-v_b}{\overline{v}-v_b}2f(v_b)[1-F(v_b)]dv_b = 0\forall v_1 \tag{B.38}$$

which is implied by Claim 4.

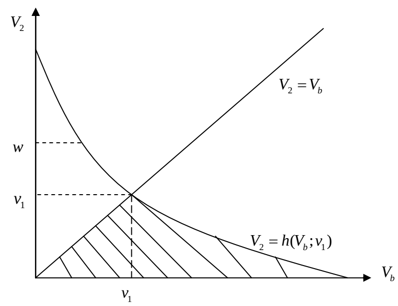

Figure B.3: Integration area for Term B.29 when $w > v_1$

Now I only need to verify the condition $G_{12}(w, v_1) > 0, \forall w, v_1 \in (\underline{v}, \overline{v}]$. Throughout the proof, keep in mind the intervals on which the Π function is defined.

First, I have

$$\frac{\partial \Pi_2(v_1; w, v_b)}{\partial v_1} = \frac{\overline{v}}{\overline{v} - v_b}(\frac{v_1}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-w}} - \frac{v_b}{\overline{v} - v_b} \geq \frac{\overline{v}}{\overline{v} - v_b}[(\frac{v_b}{\overline{v}})^{\frac{\overline{v}-w}{\overline{v}-v_b}}]^{\frac{\overline{v}-v_b}{\overline{v}-w}} - \frac{v_b}{\overline{v} - v_b} = 0,$$

the inequality holds from the fact that $\overline{v}(\frac{v_b}{\overline{v}})^{\frac{\overline{v}-w}{\overline{v}-v_b}} < v_1$.

Second,

$$\frac{\partial \Pi_3(v_1; w, v_b)}{\partial v_1} = \frac{\overline{v}(\frac{w}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-w}} - v_b}{\overline{v} - v_b} > \frac{\overline{v}(\frac{v_b}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-v_b}} - v_b}{\overline{v} - v_b} = 0,$$

where the inequality holds because of Claim 4 and $w > v_b$.

Third, for $v_1 > w$,

$$\begin{aligned}\frac{\partial \Pi_4(v_1; w, v_b)}{\partial v_1} &= \frac{\overline{v}}{\overline{v} - v_b}(\frac{v_1}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-w}} - \frac{v_b}{\overline{v} - v_b} \geq \frac{\overline{v}}{\overline{v} - v_b}(\frac{w}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-w}} - \frac{v_b}{\overline{v} - v_b} \\ &\geq \frac{\overline{v}}{\overline{v} - v_b}(\frac{v_b}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-v_b}} - \frac{v_b}{\overline{v} - v_b} \geq 0,\end{aligned}$$

where the second inequality holds because of Claim 4 and $w \geq v_b$.

Finally,

$$\frac{\partial h^{-1}(w; v_1)}{\partial v_1} = \frac{1}{\frac{\overline{v}-w}{\overline{v}-v_b}\left[1 + \frac{v_b ln(\frac{v_b}{\overline{v}})}{\overline{v}-v_b}\right]} > 0,$$

which is implied by Claim 3.

Thus, $G_{12}(w, v_1) > 0, \forall w, v_1 \in (\underline{v}, \overline{v}]$. From Theorem 1, the bidding function must satisfy

$$\begin{aligned}b^{P'}(v_1) &= G_1(v_1, v_1) = f(v_1)\left\{\int_{\underline{v}}^{v_1} v_1 \frac{\overline{v}(\frac{v_1}{\overline{v}})^{\frac{\overline{v}-v_b}{\overline{v}-v_1}} - v_b}{\overline{v} - v_b} 2f(v_b)\left[1 - F(v_b)\right] dv_b\right\} \\ &= \frac{2v_1}{(\overline{v} - \underline{v})^3}\left\{\frac{\overline{v}(\overline{v} - v_1)}{ln\frac{v_1}{\overline{v}}}\left[(\frac{v_1}{\overline{v}})^{\frac{\overline{v}-\underline{v}}{\overline{v}-v_1}} - \frac{v_1}{\overline{v}}\right] - \frac{v_1^2 - \underline{v}^2}{2}\right\}\end{aligned}$$

$$\Rightarrow b^P(v_1) = \int_{\underline{v}}^{v_1} \frac{2\xi}{(\overline{v} - \underline{v})^3} \left\{ \frac{\overline{v}(\overline{v} - \xi)}{ln\frac{\xi}{\overline{v}}} \left[(\frac{\xi}{\overline{v}})^{\frac{\overline{v} - \underline{v}}{\overline{v} - \xi}} - \frac{\xi}{\overline{v}} \right] - \frac{\xi^2 - \underline{v}^2}{2} \right\} d\xi \tag{B.39}$$

This gives us the equilibrium bidding function in the first round. Note that the second round strategy is characterized in Lemma 9. **Q.E.D**

APPENDIX C

PROOFS FOR CHAPTER 4

Proof of Lemma 12
In the final round, player B infers that player A's valuation is w, and chooses his bid in the final round accordingly, i.e. $b_B^F(w, v_B) = \frac{v_B^2 w}{(w+v_B)^2}$. Knowing player B's valuation and his response function $b_B^F(w, v_B)$, as well as his own true valuation v_A, player A responds optimally:

$$\max_{b_A^F} \; v_A \frac{b_A^F}{b_A^F + b_B^F} - b_A^F$$

where $b_B^F = \frac{v_B^2 w}{(w+v_B)^2}$.
The FOC gives us:

$$b_A^F = max\{\sqrt{v_A b_B^F} - b_B^F, 0\} = max\{\frac{v_B\sqrt{wv_A}}{w+v_B} - \frac{v_B^2 w}{(w+v_B)^2}, 0\}$$

I can summarize it as follows:

$$b_A^F(w, v_A, v_B) = \begin{cases} \frac{v_B\sqrt{wv_A}}{w+v_B} - \frac{v_B^2 w}{(w+v_B)^2} & if \; w > v_A \; and \; v_B \leq \frac{w\sqrt{v_A}}{\sqrt{w}-\sqrt{v_A}} \\ \frac{v_B\sqrt{wv_A}}{w+v_B} - \frac{v_B^2 w}{(w+v_B)^2} & if \; w \leq v_A \\ 0 & otherwise \end{cases}$$

Plugging $b_A^F(w, v_A, v_B)$ into the payoff function, I get the expected payoff:

$$\Pi_A^F(w, v_A, v_B) = \begin{cases} (\sqrt{v_A} - \frac{v_B\sqrt{w}}{w+v_B})^2 & if \; w > v_1 \; and \; v_B \leq \frac{w\sqrt{v_A}}{\sqrt{w}-\sqrt{v_A}} \\ (\sqrt{v_A} - \frac{v_B\sqrt{w}}{w+v_B})^2 & if \; w \leq v_A \\ 0 & otherwise \end{cases}$$

Q.E.D

Proof of Lemma 13
Throughout the proof, recall that I have:

- $1 < R < \sqrt[3]{4}$
- $(N-1)M\underline{v} > max\{\frac{(R-1)R^4}{2}, \frac{R^2-R}{8-4R^{\frac{3}{2}}}\}$

Here $G(w, v_1) = F(w)^{N-1}\int_{\underline{v}}^{\overline{v}}(\sqrt{v_1} - \frac{v_B\sqrt{w}}{w+v_B})^2 dF(v_B)^N$.
$G_1(v_1, v_1) > 0, \forall v_1 \in (\underline{v}, \overline{v}]$

$$\begin{aligned} \Rightarrow\quad & (N-1)F(v_1)^{N-2}f(v_1)\int_{\underline{v}}^{\overline{v}}\frac{v_1^3}{(v_1+v_B)^2}dF(v_B)^N \\ & +F(v_1)^{N-1}\int_{\underline{v}}^{\overline{v}}\frac{v_1v_B(v_1-v_B)}{(v_1+v_B)^3}dF(v_B)^N > 0, \forall v_1 \in (\underline{v}, \overline{v}] \\ \Rightarrow\quad & (N-1)f(v_1)\int_{\underline{v}}^{\overline{v}}\frac{v_1^3}{(v_1+v_B)^2}dF(v_B)^N \\ & +F(v_1)\int_{\underline{v}}^{\overline{v}}\frac{v_1v_B(v_1-v_B)}{(v_1+v_B)^3}dF(v_B)^N > 0, \forall v_1 \in (\underline{v}, \overline{v}] \end{aligned} \tag{C.1}$$

$G_{12}(w, v_1) > 0, \forall w, v_1 \in (\underline{v}, \overline{v}]$

$$\begin{aligned} \Rightarrow\quad & (N-1)F(w)^{N-2}f(w)\int_{\underline{v}}^{\overline{v}}(1 - \frac{v_B}{w+v_B}\sqrt{\frac{w}{v_1}})dF(v_B)^N \\ & +F(w)^{N-1}\int_{\underline{v}}^{\overline{v}}\frac{v_B(w-v_B)}{2\sqrt{wv_1}(w+v_B)^2}dF(v_B)^N > 0, \forall w, v_1 \in (\underline{v}, \overline{v}] \\ \Rightarrow\quad & (N-1)f(w)\int_{\underline{v}}^{\overline{v}}(1 - \frac{v_B}{w+v_B}\sqrt{\frac{w}{v_1}})dF(v_B)^N \\ & +F(w)\int_{\underline{v}}^{\overline{v}}\frac{v_B(w-v_B)}{2\sqrt{wv_1}(w+v_B)^2}dF(v_B)^N > 0, \forall w, v_1 \in (\underline{v}, \overline{v}] \end{aligned} \tag{C.2}$$

The idea is very simple. If I can prove that the items inside the integration are always greater than zero under the conditions, then I am done.

$$\begin{aligned} & (N-1)f(v_1)\frac{v_1^3}{(v_1+v_B)^2} + F(v_1)\frac{v_1v_B(v_1-v_B)}{(v_1+v_B)^3} \\ & \geq (N-1)M\frac{\underline{v}^3}{(\overline{v}+\overline{v})^2} + 1 * \frac{\overline{v}\overline{v}(\underline{v}-\overline{v})}{(\underline{v}+\underline{v})^3} \\ & = \frac{(N-1)M\underline{v}}{4R^2} + \frac{1}{8}(R^2 - R^3) \\ & = \frac{(N-1)M\underline{v} - \frac{(R-1)R^4}{2}}{4R^2} > 0 \end{aligned}$$

Thus, restriction (C.1) is satisfied.

$$\begin{aligned} & (N-1)f(w)(1 - \frac{v_B}{w+v_B}\sqrt{\frac{w}{v_1}}) + F(w)\frac{v_B(w-v_B)}{2\sqrt{wv_1}(w+v_B)^2} \\ & \geq (N-1)f(w)(1 - \frac{\overline{v}}{\underline{v}+\underline{v}}\sqrt{\frac{\overline{v}}{\underline{v}}}) + 1 * \frac{\overline{v}(\underline{v}-\overline{v})}{2\sqrt{\underline{v}\overline{v}}(\underline{v}+\underline{v})^2} \\ & = (N-1)f(w)(1 - \frac{1}{2}R^{\frac{3}{2}}) + \frac{1}{8\underline{v}}(R - R^2) \qquad recall\ that\ 1 - \frac{1}{2}R^{\frac{3}{2}} > 0 \\ & \geq (N-1)M(1 - \frac{1}{2}R^{\frac{3}{2}}) + \frac{1}{8\underline{v}}(R - R^2) \\ & = \frac{1 - \frac{1}{2}R^{\frac{3}{2}}}{\underline{v}}\Big((N-1)M\underline{v} - \frac{R^2-R}{8-4R^{\frac{3}{2}}}\Big) > 0 \end{aligned}$$

Thus, restriction (C.2) is satisfied.
Q.E.D

Proof for Proposition 12
I first consider the following all-pay auction problem.

$$\max_{w}\ F(w)^{N-1}\int_{\underline{v}}^{\overline{v}}(\sqrt{v_1}-\frac{v_B\sqrt{w}}{w+v_B})^2dF(v_B)^N-b^P(w) \tag{C.3}$$

Since $G_1(v_1,v_1)>0$ and $G_{12}(w,v_1)>0$, Theorem 1 tells us that the equilibrium bidding function is:

$$\begin{aligned} b^P(v_1) &= \int_{\underline{v}}^{v_1}\left(\int_{\underline{v}}^{\overline{v}}\frac{\xi^3}{(\xi+v_B)^2}dF(v_B)^N\right)dF(\xi)^{N-1} \\ &+\int_{\underline{v}}^{v_1}\left(F(\xi)^{N-1}\int_{\underline{v}}^{\overline{v}}\frac{\xi v_B(\xi-v_B)}{(\xi+v_B)^3}dF(v_B)^N\right)d\xi \end{aligned} \tag{C.4}$$

Now, I move on to prove that under the restrictions above, the bidding strategy (C.4) consists of an equilibrium in the original model.

Necessary condition: if player 1 over-represents his valuation but just deviates locally, then $\frac{w\sqrt{v_1}}{\sqrt{w}-\sqrt{v_1}}$ goes to infinity. Thus, if player 1 under-represents or over-represents locally, his problem at the beginning of the preliminary round is exactly the same as problem (C.3) above, so the necessary condition is already verified.

Sufficient condition: under the restrictions above, it is optimal to truthfully represent the valuation in problem (C.3). Thus, I have:

$$\begin{aligned} &v_1\ F(v_1)^{N-1}\int_{\underline{v}}^{\overline{v}}(\sqrt{v_1}-\tfrac{v_B\sqrt{v_1}}{v_1+v_B})^2dF(v_B)^N-b^P(v_1) \\ &> w\ F(v_1)^{N-1}\int_{\underline{v}}^{\overline{v}}(\sqrt{v_1}-\tfrac{v_B\sqrt{w}}{w+v_B})^2dF(v_B)^N-b^P(w),\quad \forall w\in[\underline{v},\overline{v}], w\neq v_1 \end{aligned} \tag{C.5}$$

If player 1 under-represents his valuation, from (C.5), then the payoff is lower.
If player 1 over-represents his valuation, then

$$\begin{aligned} &v_1\ F(v_1)^{N-1}\int_{\underline{v}}^{\overline{v}}(\sqrt{v_1}-\tfrac{v_B\sqrt{v_1}}{v_1+v_B})^2dF(v_B)^N-b^P(v_1) \\ &> w\ F(v_1)^{N-1}\int_{\underline{v}}^{\overline{v}}(\sqrt{v_1}-\tfrac{v_B\sqrt{w}}{w+v_B})^2dF(v_B)^N-b^P(w) \\ &\geq w\ F(v_1)^{N-1}\int_{\underline{v}}^{min\{\overline{v},\ \frac{w\sqrt{v_1}}{\sqrt{w}-\sqrt{v_1}}\}}(\sqrt{v_1}-\tfrac{v_B\sqrt{w}}{w+v_B})^2dF(v_B)^N-b^P(w)\quad \forall w>v_1 \end{aligned} \tag{C.6}$$

Thus, from (C.6), it also decreases the payoff.
Hence, it is optimal to represent $w=v_1$.
Q.E.D

Proof for Proposition 13
Denote $T(v)=\int_{\underline{v}}^{\overline{v}}\frac{\zeta(v-\zeta)}{(v+\zeta)^3}dF(\zeta)^N$, then $S(v)=F(v)^{N-1}vT(v)$.
It is not hard to see that $T(\underline{v})<0$ and $T(\overline{v})>0$. Meanwhile,

$$\begin{aligned} T'(v) &= \int_{\underline{v}}^{\overline{v}}\frac{2\zeta(2\zeta-v)}{(v+\zeta)^4}dF(\zeta)^N \\ &\geq \int_{\underline{v}}^{\overline{v}}\frac{2\zeta(2\underline{v}-\overline{v})}{(v+\zeta)^4}dF(\zeta)^N \\ &= \underline{v}(2-R)\int_{\underline{v}}^{\overline{v}}\frac{2\zeta}{(v+\zeta)^4}dF(\zeta)^N>0 \end{aligned}$$

The last step follows the condition that $R < 2$.

Thus, $T(v)$ is an strictly increasing function as well as $T(\underline{v}) < 0$ and $T(\overline{v}) > 0$. So there exists a threshold $\underline{v} < v^* < \overline{v}$, such that for $\underline{v} \leq v < v^*$, $T(v) < 0$; and for $\overline{v} \geq v > v^*$, $T(v) > 0$.

For $v = \underline{v}$, $S(\underline{v}) = 0$, so the marginal willingness to bid is the same.

For $\underline{v} < v < v^*$, $S(v) = F(v)^{N-1}vT(v) < 0$, the marginal willingness to bid is lower in the presence of the signaling effect.

For $v^* < v \leq \overline{v}$, $S(v) = F(v)^{N-1}vT(v) > 0$, the marginal willingness to bid is higher in the presence of the signaling effect.

Q.E.D